VOCALOID

TABLE OF CONTENTS

TOPIC 1: LIMITS OF FUNCTIONS

Limit Definition: a value that f(x) approaches as the values of x approach a specific number
Limit Notation:
$\lim_{x \to a} f(x) = L$, read as: **the limit of f(x) as x approaches a is L**

Calculating Limits using Algebra, Tables, and Graphs:
The most straightforward way to find a limit is to **substitute the specified x value into the function**. The resulting y value is the limit.
Example:

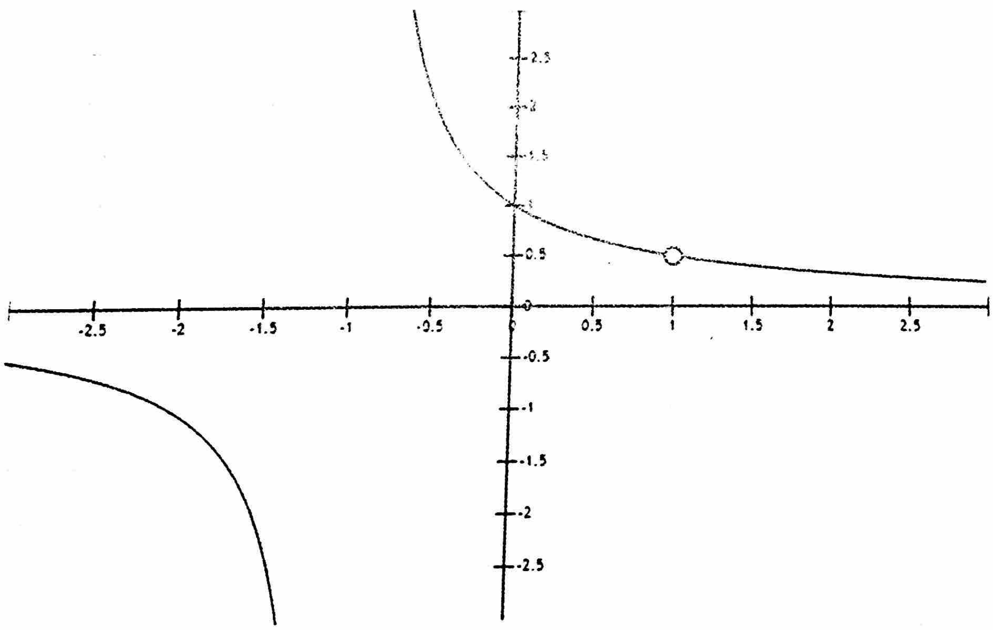
$$\lim_{x \to 1} (x - 1) = ?$$

When x =1, (x-1) = 0. Therefore, the limit as x approaches 1 is 0.
When the function is undefined at the specified value, however, we can use a **graph, a table, or algebra** to find the limit.
Example:

$$\lim_{x \to 1} \frac{x - 1}{x^2 - 1} = ?$$

When x =1, $\frac{x-1}{x^2-1} = \frac{1-1}{1-1} = \frac{0}{0}$ (which is an indeterminate form)

<u>With a Graph:</u>
We can see that f(x) is approaching a y-value of 0.5 as x approaches 1 from either side, even though f(1) is not defined. Therefore, the limit of f(x) as x approaches 1 is 0.5.



With a Table:
Instead of graphing the function, we can construct a table of x values and their corresponding y values. Take x values slightly above and slightly below. On a graphing calculator such as the TI-84, set the TblStart value as your x value and set the Δ Tbl value as 0.001.

x	y
.997	.50075
.998	.5005
.999	.50025
1.000	UNDEFINED
1.001	.49975
1.002	.4995
1.003	.49925

It appears that the limit is approximately 0.5, or $\frac{1}{2}$.

With Algebra:
We can simplify the denominator, if we recognize that $(x^2 - 1)$ is a difference of squares.

$$\lim_{x\to 1} \frac{x-1}{x^2-1} = ?$$

$$\frac{x-1}{x^2-1} = \frac{x-1}{(x+1)(x-1)} = \frac{1}{x+1}$$

Now, we can substitute x = 1 into the function.

$$\frac{1}{x+1} = \frac{1}{2}$$

In some situations, we can multiply either the denominator or the numerator by its **conjugate**.
Important: Whatever operation performed on the denominator must also be performed on the numerator (or vice versa) in order to preserve the function's identity.

$$\lim_{x\to 4} \frac{\sqrt{x} - 2}{x-4} = ?$$

The conjugate of $(\sqrt{x}) - 2$ is $(\sqrt{x}) + 2$

$$\frac{\sqrt{x}-2}{x-4} = \frac{(\sqrt{x}-2)(\sqrt{x}+2)}{(x-4)(\sqrt{x}+2)} = \frac{x-4}{(x-4)(\sqrt{x}+2)} = \frac{1}{(\sqrt{x})+2} = \frac{1}{(\sqrt{4})+2} = \frac{1}{4}$$

Limits that Fail to Exist & One-Sided Limits:

Limit Existence Theorem: If each one sided limit for a particular x value is the same number, then the corresponding two-sided limit is that same number

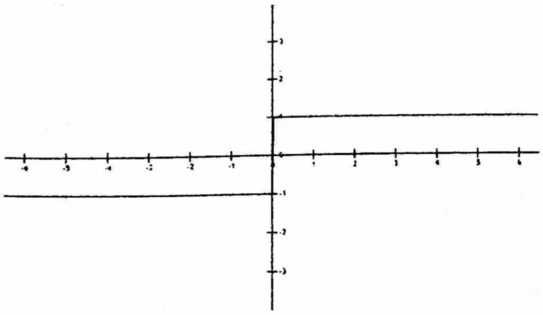
$$\lim_{x\to a} f(x) = L \text{ if and only if } \lim_{x\to a^+} f(x) = L \text{ and } \lim_{x\to a^-} f(x) = L$$

These are three situations in which a limit will fail to exist.

1. **f(x) approaches a different value from the right than from the left**
2. **f(x) increases or decreases without bound as x approaches a**
3. **f(x) oscillates between 2 fixed values**

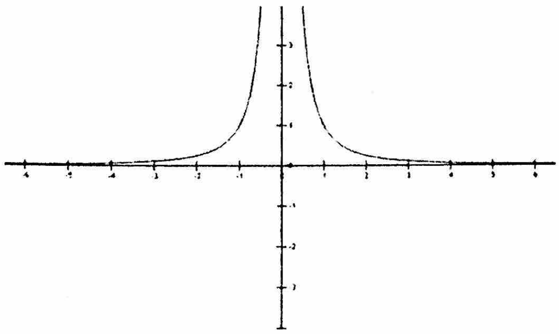
Example 1:

$\lim_{x\to 0} \frac{|x|}{x}$



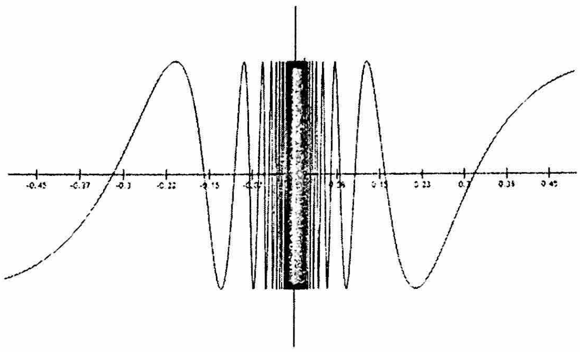
Example 2:

$\lim_{x\to 0} \frac{1}{x^2}$



Example 3:

$\lim_{x\to 0} \sin(\frac{1}{x})$



<u>One-Sided Limits:</u>

We can examine the behavior of a function as it approaches a particular x value from one side. L can be a number, positive infinity, or negative infinity.

<u>One-Sided Limit Notation:</u>

$$\lim_{x\to a+} f(x) = L$$

read as: **the limit of f(x) as x approaches a from the right is L**

$$\lim_{x\to a-} f(x) = L$$

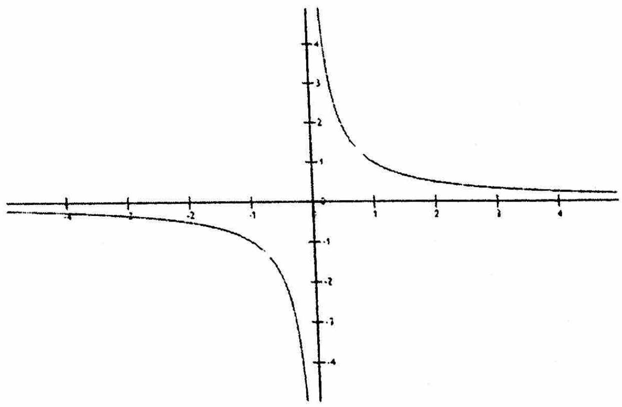
read as: **the limit of f(x) as x approaches a from the left is L**

Vertical and Horizontal Asymptotes:

If $\lim_{x\to a+} f(x) = \pm\infty$ OR $\lim_{x\to a+} f(x) = \pm\infty$ then there is a vertical asymptote at x = a

Example: f(x) = $\frac{1}{x}$

If we examine the graph, we can see that f(x) approaches positive infinity from the right of zero and negative infinity from the left of zero. We can substitute values above and values below x to examine how the function is behaving.



x	y
-0.003	-333.3
-0.002	-500
-0.001	-1000
0	UNDEFINED
0.001	1000
0.002	500
0.003	333.3

<u>Horizontal Asymptotes:</u>

If $\lim_{x\to\infty} f(x) = b$ OR $\lim_{x\to-\infty} f(x) = b$ then there is a horizontal asymptote at y = b

For rational functions, let **m** be the degree of the highest degree term in the numerator and let **n** be the degree of the highest degree term in the denominator:

1. **If m = n, then the horizontal asymptote is the ratio of the coefficients of the highest degree terms (numerator coefficient divided by denominator coefficient)**
2. **If m < n, then the horizontal asymptote is y = 0.**
3. **If m > n, then there is no horizontal asymptote**

Example 1:

$\lim_{x\to\infty} f(x) = \frac{2x^2+x+1}{x^2} = \frac{2}{1} = 2$ (horizontal asymptote at y = 2)

Example 2:

$\lim_{x\to\infty} f(x) = \frac{2x+x+1}{x^2} = 0$ (horizontal asymptote at y = 0)

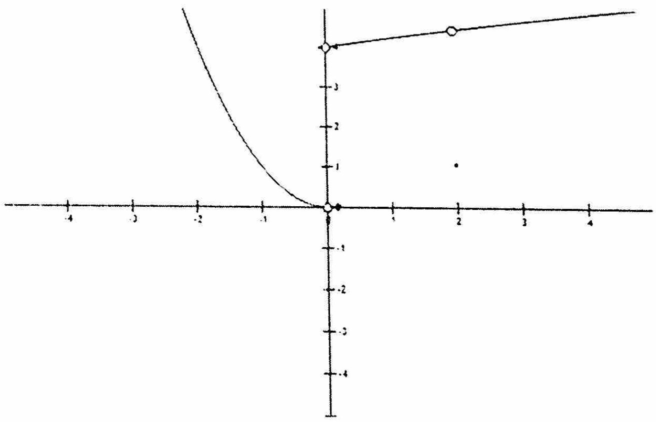
Example 3:

$\lim_{x\to\infty} f(x) = \frac{2x^2+x+1}{x} = \infty$ (no horizontal asymptote)

Limits from Piecewise Functions:

Using the function definition, sketch a graph of the function and examine the function using the techniques discussed in this topic.

$$f(x) = \begin{cases} x^2 & (-\infty, 0) \\ \sqrt{(x+4)} + 2 & (0,2) \cup (2, \infty) \\ 2 & \{2\} \end{cases}$$



$\lim_{x\to0+} f(x) = 4$	f(0) is undefined	$\lim_{x\to2+} f(x) = 4.5$	$\lim_{x\to2} f(x) = 4.5$
$\lim_{x\to0-} f(x) = 0$	f(2) = 1	$\lim_{x\to2-} f(x) = 4.5$	**Note that the limit of a function is not always the value of the function!**

TOPIC 2: CONTINUITY

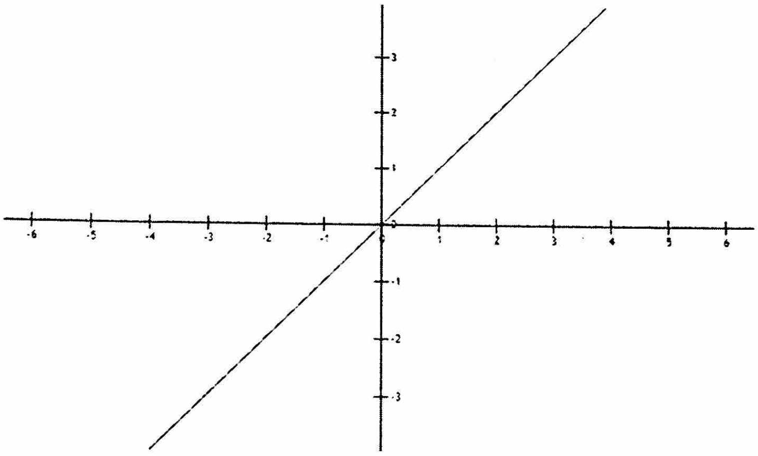
When is a function continuous at x = a?

f(x) is continuous at x = a if:

1. $\lim_{x \to a} f(x)$ exists [roads must meet]
2. f(x) is defined [bridge must exist]
3. $\lim_{x \to a+} f(x) = f(a)$ [roads must meet at bridge]

Example: f(x) = x is continuous at x = 0 because:

1. $\lim_{x \to 0} f(x)$ exists [f(x) approaches 0 from both sides]
2. f(0) is defined
3. $\lim_{x \to 0} f(x)$ = f(0) = 0



Discontinuity (Types):

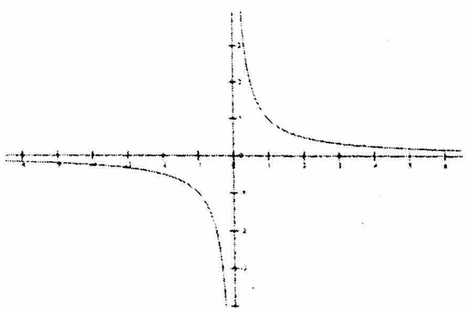
1. **Infinite:** either one sided limit is $+\infty$ or $-\infty$ (vertical asymptote)
2. **Jump:** the limit approaches different values from either side
3. **Point:** the limit of the function is not the value of the function

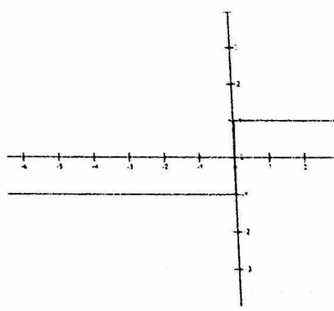
Example 1: Infinite Discontinuity (Left)

f(x) = $\frac{1}{x}$ is not continuous at x = 0 because

$\lim_{x \to 0+} f(x)$ = +∞

$\lim_{x \to 0-} f(x)$ = -∞





Example 2: Jump Discontinuity (Right)

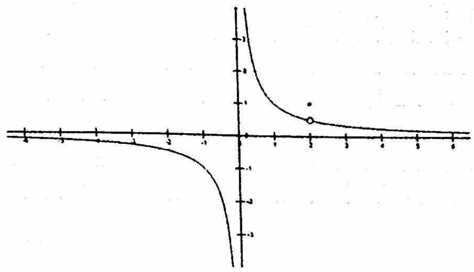
F(x) = $\frac{|x|}{x}$ is not continuous at x = 0 because

$\lim_{x \to 0+} f(x)$ = 1

$\lim_{x \to 0-} f(x)$ = -1

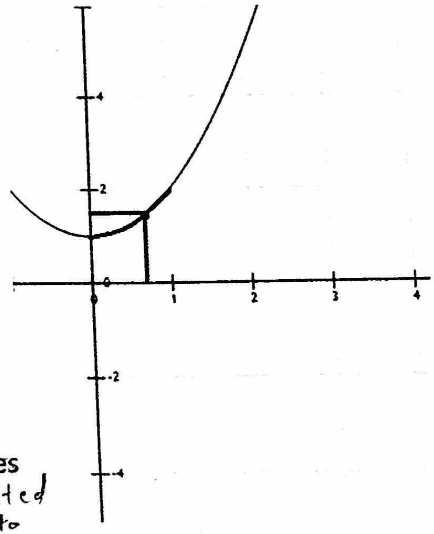
Example 3: Point Discontinuity

This function is not continuous at x = 2 because $\lim_{x\to 2} f(x) \neq f(2)$



Intermediate Value Theorem: if f(x) is continuous on [a, b] then:
If μ is a value such that $f(a) \le \mu \le f(b)$ then there is at least one number c between a and b such that f(c) = μ

Example: $f(x) = x^2 + 1$ is continuous on [0, 1]
f(0) = 1
f(1) = 2
Let μ be 1.5 (1<1.5<2)
There must be at least one number c between 0 and 1 such that f(c) = 1.5



We can find c by setting 1.5 equal to f(c). and solving for c.
f(c) = 1.5
$x^2 + 1 = 1.5$
$x^2 = .5$
$x = (\sqrt{2})/2 \approx 0.707$ (note that we only consider the positive value because c is ~~res~~ limited to [0,1]

Extension: If x_1 and x_2 are numbers between a and b, and if $f(x_1)$ and $f(x_2)$ have opposite signs, then there is at least one value of #c between x_1 and x_2 such that f(c) = 0

VOCALOID

TOPIC 3: DIFFERENTIAL CALCULUS

Derivative of a Function: the slope of the line tangent to f(x) at any value x
To find the derivative of a function, we can use the limit definition below:

Limit Definition of the Derivative of a Function: the instantaneous rate of change of f(x), defined as the limit of the average rate of change as the interval of average change approaches zero (also notated as: f′(x), y′,or $\frac{dy}{dx}$)

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

or

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

Example: $f(x) = x^3$

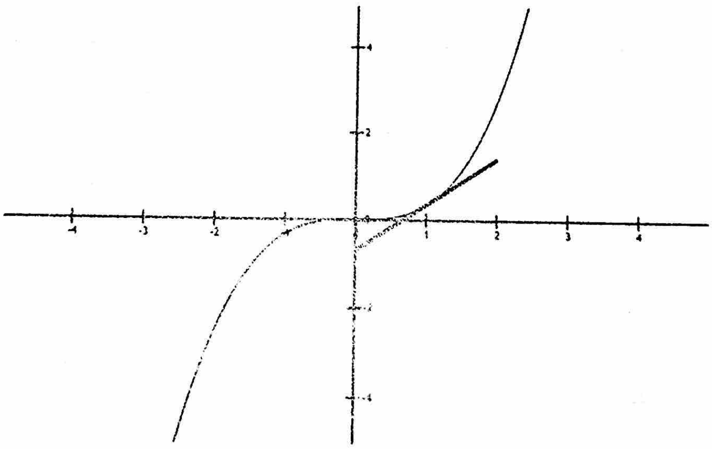
$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^3-x^3}{h} = \lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}$$

$$= \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h} = \lim_{h\to 0} 3x^2+3xh+h^2 = 3x^2$$

Derivative of a Function at x = a: the slope of the tangent line drawn at x = a

Example: $f(x) = \frac{1}{3}(x^3)$, find the derivative at x = 1

If we draw a tangent line at x = 1, we see that the slope of the tangent line is 1. The derivative of f(x) at x = 1 is 1.



We can also use the limit definition to find the derivative of the function, then substitute x = 1 into the derivative.

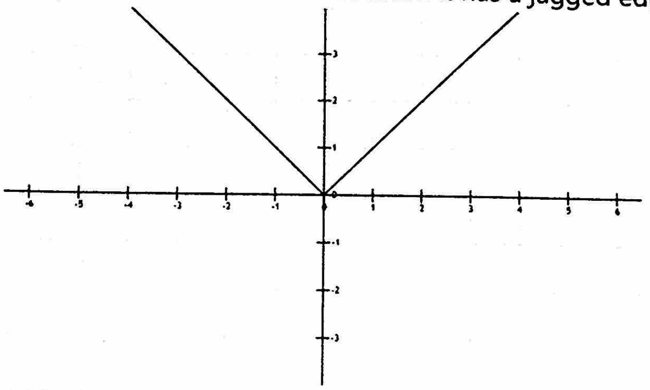
$$F'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(\frac{1}{3})(x+h)^3-\frac{1}{3}(x)^3}{h} = \lim_{h\to 0}\frac{\frac{1}{3}(x^3+3x^2h+3xh^2+h^3)-\frac{1}{3}x^3}{h}$$

$$= \lim_{h\to 0}\frac{\frac{1}{3}x^3+x^2h+xh^2+\frac{1}{3}h^3-\frac{1}{3}x^3}{h} = \lim_{h\to 0}\frac{x^2h+xh^2+\frac{1}{3}h^3}{h} = \lim_{h\to 0} x^2+xh+\frac{1}{3}h^2 = x^2 = (1)^2 = 1$$

Relationship between Diffrentiability and Continuity: If a function is not continuous at x = a, it is not diffrentiable at x = a. Being continuous at x = a does not guarantee differentiability.

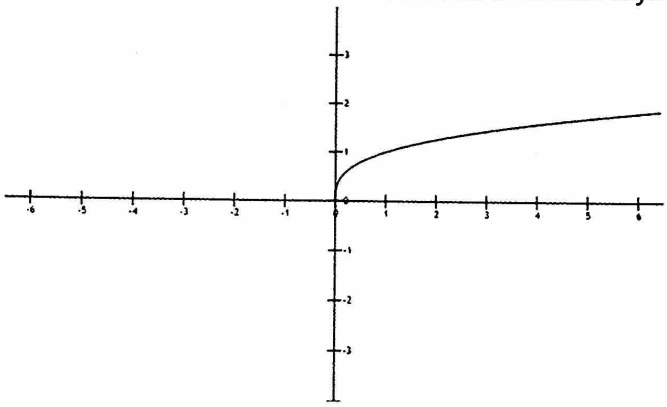
Example 1: (Jagged Edge)

F(x) = |x| is continuous at x = 0, but not differentiable because it has a jagged edge at x = 0.



Example 2: (Vertical Tangent Line)

f(x) = $x^{\frac{1}{3}}$ is continous at x = 0, but not diffrentiable because it has a vertical asymptote at x = 0.



Constant Rule, Power Rule, Product/Quotient Rule, Chain Rule: Diffrentiate each term separately!

Constant Rule: if f(x) = a, a ε ℝ, then f′(x) = 0

Example: f(x) = *e*, f′(x) = 0

Constant Multiple Rule: if f(x) = ax, a ε ℝ, then f′(x) = a

Example: f(x) = 2x + 1, f′(x) = 2

Power Rule: if f(x) = x^n, n=Q, then f′(x) = nx^{n-1}. if f(x) = ax^n, then f′(x) = anx^{n-1}

Example 1: f(x) = x^4, f′(x) = $4x^3$

Example 2: f(x) = 2x^3, f′(x)= $2(3)x^2 = 6x^2$

Product Rule: $\frac{d}{dx}(u \cdot v) = v \cdot u' + u \cdot v'$

Example: f(x) = $(x^4)(x+1) = (4x^3)(x+1)+(1)(x^4)$

Quotient Rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \cdot u' - u \cdot v'}{v \cdot v}$

Example: f(x) = $\frac{x^2+1}{x} = \frac{x(2x)-(x^2+1)(1)}{x^2}$

Chain Rule: $\frac{d}{dx}[f(g(x))]$ = f′(g(x)) · g′(x)

Example: f(x) = x^5, g(x) = $\frac{1}{2}$x

$\frac{d}{dx}[f(g(x))]$ = f′(g(x)) · g′(x) = $5(\frac{1}{2}x)^4 \cdot \frac{1}{2}$

Tangent Lines and Normal Lines:

As stated before, the derivative of f(x) at x = a is the slope of the tangent line at x = a. Using the point-slope formula, we can write the equation of the tangent line.

Example: Find the tangent line of f(x) at x = 2

$f(x) = \sqrt{(x^2-3)}$, so $\frac{dy}{dx} = \frac{1}{2}(x^2-3)^{-\frac{1}{2}}(2x) = \frac{1}{2}(2^2-1)^{-\frac{1}{2}}(4) = 2$

point-slope formula: $y - y_1 = m(x-x_1)$

From the equation, we know that the tangent line will intersect the graph at (2,1). Substitute 2 for x_1 and 1 for y_1.

$y - y_1 = m(x-x_1) = y - 1 = 2(x-2)$

To find the normal line, find the derivative of the function as before, then take the negative reciprocal. That value is the slope of the normal line. The values of x_1 and y_1 will be calculated the same as before, since the normal line also crosses (x_1, y_1).

Higher Order Derivatives:

f'(x) is the first derivative of f(x)

f''(x) is the second derivative of f(x), and it is found by taking the derivative of f'(x)

f'''(x) is the third derivative of f(x), and it is found by taking the derivative of f''(x)

... and so on.

Example:

$f(x) = x^3+2x+1$

$f'(x) = 3x^2 + 2$

$f''(x) = 6x$

$f'''(x) = 6$

Implicit Differentiation: For more complex functions, we can differentiate implicitly.

Example: Differentiate $x^2 + y^2 = 1$ with respect to x

Differentiate each term. Note that the chain rule applies for x^2 and y^2.

$x^2 + y^2 = 1$

$(2x)\frac{dx}{dx} + (2y)\frac{dy}{dx} = 0$

$\frac{dx}{dx} = 1$, so:

$(2x) + (2y)\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-x}{y}$

First Derivative Test:

1. If f'(x) > 0, then f(x) is increasing
2. If f'(x) < 0, then f(x) is decreasing
3. If f'(x) = 0, then f(x) is constant

Critical Values: x values where f'(x) = 0 or f'(x) is undefined

Relative Minima/Maxima: If x = c is in the domain, then:

1. If f'(c) changes from – to +, f(c) is a relative minimum
2. If f'(c) changes from + to –, f(c) is a relative maximum

Given a function, be able to find:

1. The domain
2. The x-coordinate(s) of relative extrema (critical values)
3. Intervals where the function is increasing or decreasing

Example: $f(x) = x^4 - x^2$

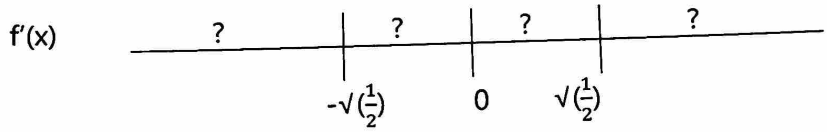
$f'(x) = 4x^3 - 2x = 2x(2x^2-1)$

$f'(x) = 0$ when $x = 0$ and when $x = \pm\sqrt{(\frac{1}{2})}$ (set $(2x^2-1) = 0$ and solve for x)

So:

1. The domain is $\mathbb{R}$
2. The critical values are: $x=0$, $x = \sqrt{(\frac{1}{2})}$, and $x = -\sqrt{(\frac{1}{2})}$

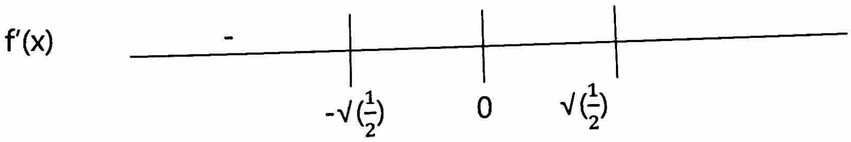
We can use a sign chart to assist us in finding when f(x) is increasing or decreasing. Plot the critical values on a number line.



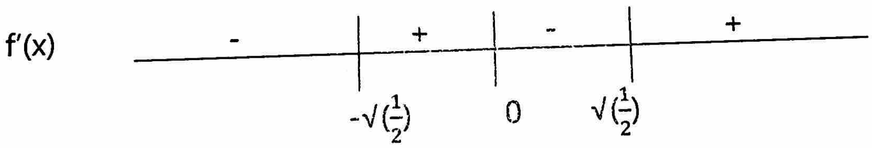
Determine the behavior of the function one "segment" at a time. For example, substitute an x value that is less than $-\sqrt{(\frac{1}{2})}$. -1 works well here.

$f'(x) = 4x^3 - 2x = 2x(2x^2-1) = (-2)(2-1) = -2$ (negative, so the function is decreasing between (-infinity, $-\sqrt{(\frac{1}{2})}$)

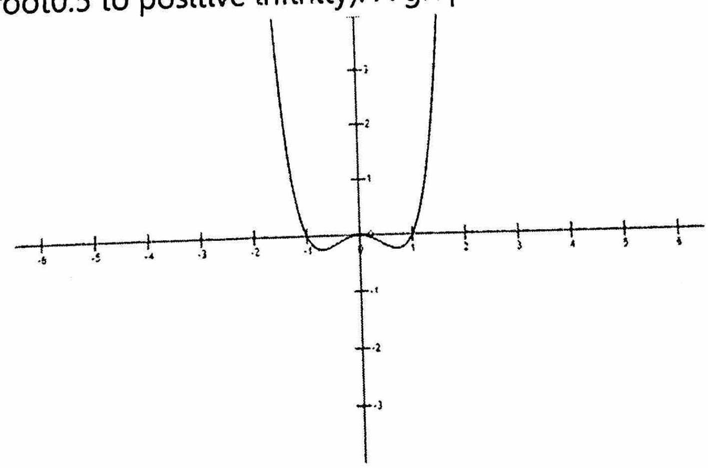
Put a negative sign in the corresponding place.



Continue this process until the chart is complete.



So, we conclude that f(x) is decreasing on the intervals (-infinity to –root0.5) and increasing on the intervals (-root0.5 to 0 and root0.5 to positive infinity). A graph of the function confirms our results:



Second Derivative Test: If a is a critical value, then:

1. If $f''(a) > 0$, then $x = a$ is a relative min
2. If $f''(a) < 0$, then $x = a$ is a relative max
3. If $f'''(a) = 0$, then the second derivative test is inconclusive

+ + for relative min
$\cup$

- - for relative max
$\cap$

0 0 for inconclusive
--

Let the eyes be the sign of the second derivative, and let the mouths be the behavior of the function.

Example: confirm that $f(x) = x^4 - x^2$ has a relative minimum at $x = (\sqrt{(\frac{1}{2})})$

$f'(x) = 4x^3 - 2x$

$f''(x) = 12x^2 - 2$

$f''(\sqrt{(\frac{1}{2})}) = 12(\frac{1}{2}) - 2 = 4$ so there is a relative minimum at $x = (\sqrt{(\frac{1}{2})})$

<u>Concavity:</u>

1. If $f''(x) > 0$ then $f(x)$ is concave up
2. If $f''(x) < 0$, then $f(x)$ is concave down

<u>Points of inflection</u>: if $x = c$ is in the domain, then:
if there is a sign change at $f''(c)$, then $(c, f(c))$ is a point of inflection

Given a function, be able to find intervals of concavity and points of inflection (use a sign chart for $f''(x)$)

Example: $f(x) = x^4-4x^3$

Domain is $\mathbb{R}$

$f'(x) = 4x^3-12x^2 = 4x^2(x-3)$

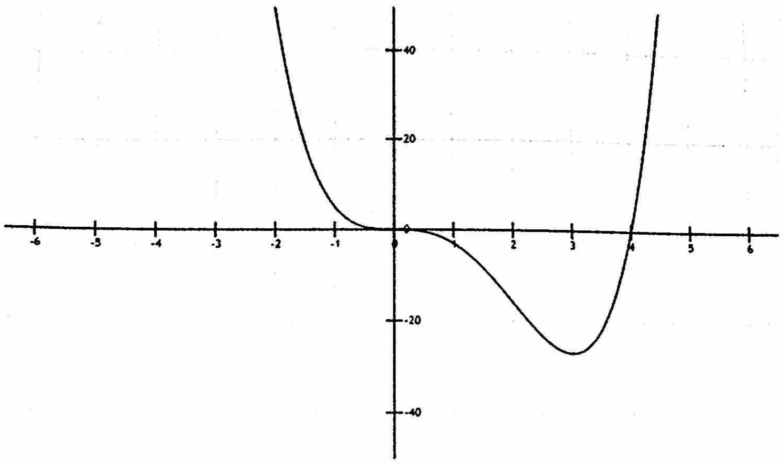
$f''(x) = 12x^2-24x$

$f''(x) = 0$ when $x = 0$ and $x = 2$

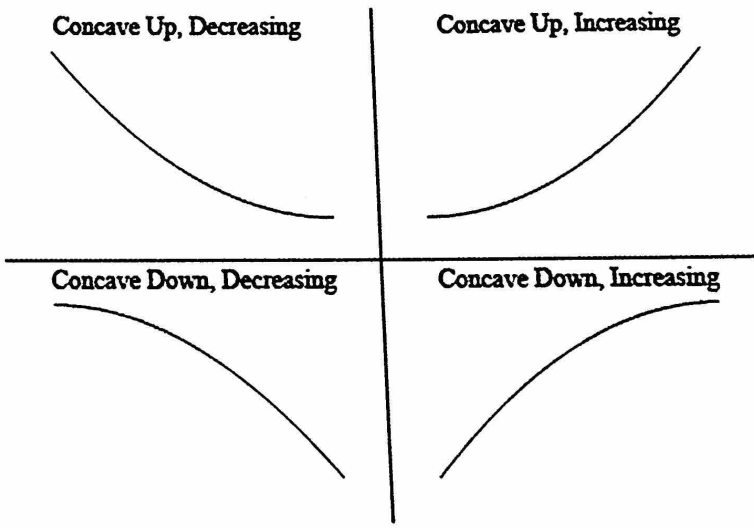
Complete the sign chart: (x values of -1, 1, and 3 work well here)

$f''(x)$	+	|	-	|	+
		0		2	

Therefore, the graph is concave up on (-infinity, 0) and (2, +infinity) and concave down (0,2).
To find the points of inflection, simply substitute $x = 0$ and $x = 2$ into the original function.
(0,0) and (2,-16)



Given a Graph of f′(x), Sketching f(x): Using the first derivative sign chart and the second derivative sign chart, we can sketch a possible graph of f(x) if we know f′(x)

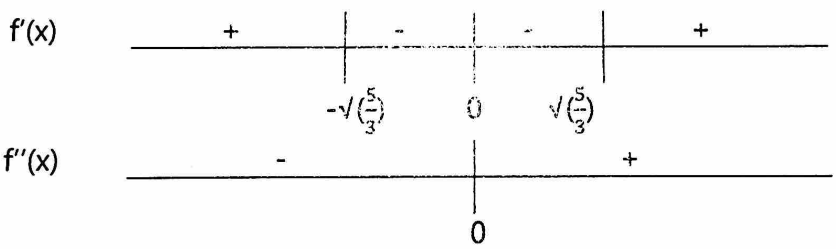


Example: $f'(x) = 3x^2-5$

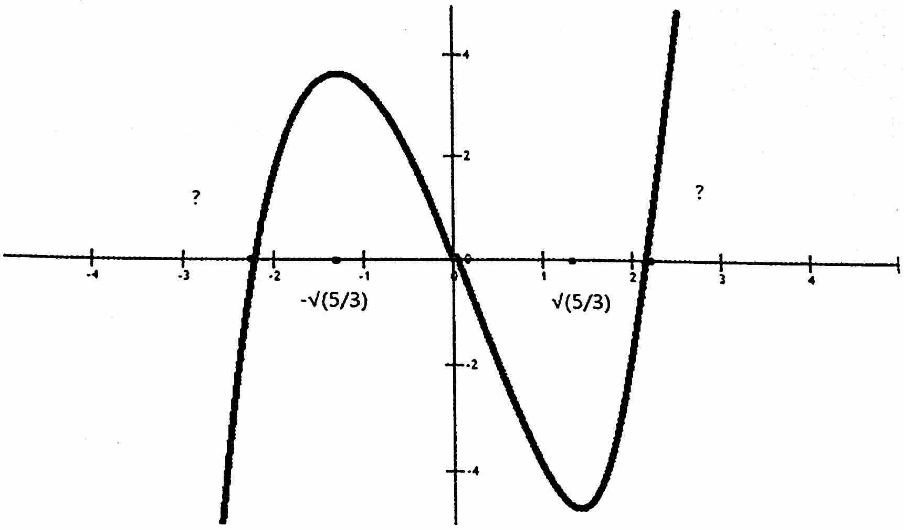
$f'(x) = 3x^2-5$
$f''(x) = 6x$

Domain is $\mathbb{R}$

$f'(x) = 0$ when $x = \sqrt{(\frac{5}{3})}$ and $x = -\sqrt{(\frac{5}{3})}$
$f''(x) = 0$ when $x = 0$
(if the graph of f′(x) is provided, we can determine these values graphically)
Make a sign chart for f′(x) and a sign chart for f″(x). Line them up to make sketching easier.



On the interval $(-\infty, -\sqrt{(\frac{5}{3})})$ the function is increasing and concave down
On the interval $(-\sqrt{(\frac{5}{3})}, 0)$ the function is decreasing and concave down
On the interval $(0, \sqrt{(\frac{5}{3})})$ the function is decreasing and concave up
On the interval $(\sqrt{(\frac{5}{3})}), \infty)$ the function is increasing and concave up



Note: Since we are not given the actual function, we do not know the precise location of the zeros. We would need the actual function to find the zeros.

Loagarithms, Trigonometric Functions, and Inverse Trigonometric Functions:
General Definition: if y = b^x then $\log_b(y) = x$

$f(x) = \ln x$, $f'(x) = \frac{1}{x}$

$f(x) = \ln(g(x))$, $f'(x) = \frac{g'(x)}{g(x)}$

Example: $f(x) = \ln x^3 + 1$, $f'(x) = \frac{3x^2 2}{x^3+1}$

$f(x) = e^x$, $f'(x) = e^x$

$f(x) = e^{g(x)}$, $f'(x) = g'(x) \cdot e^{g(x)}$

Example: $f(x) = e^{x^3+1}$, $f'(x) = (3x^2 2)e^{x^3+1}$

$f(x) = a^{g(x)}$ where a is a positive real number

$f'(x) = g'(x) \cdot a^{g(x)} \cdot \ln a$

Example: $f(x) = 2^{x^3+1}$, $f'(x) = \ln 2(3x^2)2^{x^3+1} = (x^2)(\ln 8)(2^{x^3+1})$

$f(x) = \log_a(h(x))$, $a>0$

$f'(x) = \frac{h'(x)}{h(x)} \cdot \frac{1}{\ln a}$

$\frac{1}{\ln a} = \log_a e$

Example: $f(x) = \log_{10}(e^x)$, $f'(x) = \frac{e^x}{e^x} \cdot \frac{1}{\ln 10} = \log_{10}(e) = \log(e)$

<u>Trigonometric Derivatives:</u>

sin′(x) = cos(x)
sin′(g(x)) = g′(x)cos(x)

Example: sin′(x^3 + 1) = ($3x^2$)cos(x^3 + 1)

cos′(x) = -sin(x)
cos′(g(x)) = -g′(x)sin(x)

Example: cos′(x^3 + 1) = -($3x^2$)sin(x^3 + 1)

tan′(x) = $\sec^2$(x)
tan′(g(x)) = g′(x)$\sec^2$(x)

Example: tan′(x^3 + 1) = ($3x^2$) $\sec^2$(x^3 + 1)

cot′(x) = -$\csc^2$(x)
cot′(g(x)) = -g′(x)$\csc^2$(x)

Example: cot′(x^3 + 1) = -($3x^2$)$\csc^2$(x^3 + 1)

sec′(x) = sec(x)tan(x)
sec′(g(x)) = g′(x)sec(x)tan(x)

Example: sec′(x^3 + 1) = ($3x^2$) sec(x^3 + 1)tan(x^3 + 1)

csc′(x) = -csc(x)cot(x)
csc′(g(x)) = -g′(x)csc(x)cot(x)

Example: csc′(x^3 + 1) = -($3x^2$)csc(x^3 + 1)cot(x^3 + 1)

<u>Inverse Trigonometric Derivatives:</u>

$(\sin^{-1}(x))'$ = arcsin′(x) = $\frac{x'}{\sqrt{(1-x^2)}}$

Example: arcsin′(x^3) = $\frac{3x^2}{\sqrt{(1-x^6)}}$

$(\cos^{-1}(x))'$ = arcos′(x) = $-\frac{x'}{\sqrt{(1-x^2)}}$

Example: arccos′(x^3) = $-\frac{3x^2}{\sqrt{(1-x^6)}}$

$(\tan^{-1}(x))'$ = arctan(x) = $\frac{1}{1+x^2}$

Example: arctan′(x^3) = $\frac{3x^2}{1+x^6}$

L'Hopital's Rule:

$\lim_{x\to c}\left(\frac{f(x)}{g(x)}\right) = \lim_{x\to c}\left(\frac{f'(x)}{g'(x)}\right)$ when taking the limit gives ($\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$)

Example:
$\lim_{x\to\infty}\left(\frac{x^2}{e^x}\right) = \frac{\infty}{\infty}$, so $\lim_{x\to\infty}\left(\frac{2x}{e^x}\right) = \lim_{x\to\infty}\left(\frac{2}{e^x}\right) = 0$ so $\lim_{x\to\infty}\left(\frac{x^2}{e^x}\right) = 0$

Logarithmic Differentiation: (variable raised to a variable)
Example: $y = x^x$

$\ln y = \ln x^x = x\ln x$

take the derivative of each side:

$y'\frac{1}{y} = (x)\frac{1}{x} + \ln x = 1 + \ln x$

multiply both sides by y:

$y' = y(1+\ln x) = x^x(1+ \ln x)$

<u>Indeterminate Form:</u>

$\frac{0}{0}, \frac{\pm\infty}{\pm\infty}, \infty-\infty, 0\cdot\infty, 0^0, \infty^0$

Derivative of an Inverse:

$(f^{-1})'(d) = \frac{1}{f'(c)}$ where $f(c) = d$

Example: $f(x) = \frac{1}{4}x^3+x-1$, find $(f^{-1})'(3)$

solve for $f(c) = d$, where $d = 3$
$\frac{1}{4}x^3+x-1=3,\ x=2$
evaluate $\frac{1}{f'(c)}$
$f'(x) = \frac{3}{4}x^2+1$
$\frac{1}{f'(x)} = \frac{1}{\frac{3}{4}x^2+1} = \frac{1}{\frac{3}{4}2^2+1} = \frac{1}{4}$

Extreme Value Theorem: if f(x) is continuous on a closed interval [a,b], then f has both an absolute maximum and minimum. If f has one and only one relative extreme, it must also be absolute

Mean Value Theorem & Rolle's Theorem:
Mean Value Theorem:
if f(x) is continuous on [a,b] and differentiable on (a,b) then there exists a number c in (a,b) such that

$f'(c) = \frac{f(b)-f(a)}{b-a}$

Example:

f(x) = √(x-2) (2,6)

$f'(x) = \frac{1}{2}(x\text{-}2)^{(0.5)} = \frac{1}{2(x-2)^{\frac{1}{2}}}$

$\frac{f(6) - f(2)}{(6-2)} = 0.5$

0.5 = √(x-2), x =3

Rolle's Theorem: if f(x) is continuous on [a,b] and differentiable on (a,b), then if f(a) = f(b) then there is at least one number on (a,b) such that f'(c) = 0

Example: f(x) = x^4-2x^2 (-2,2)

F(-2) = f(2) so:

F'(x) = 4x^3 -4x = 4(x^2-1) = 0, x = 1, -1, and 0

Absolute Extrema: to find absolute extrema: 1) find critical values 2) evaluate f at critical numbers 3) evaluate f at endpoints

Example: $f(x) = 2x - 3x^{(\frac{2}{3})}$ [-1,3]

$f'(x) = 2 - x^{(\frac{-1}{3})}$ critical values: x = 0, x = 1

f(-1) = -5

$f(3) = 6-(3)(3)^{(\frac{2}{3})}$

f(1) = -1

f(0)= 0

Therefore, the absolute minimum is (-1, -5) and the absolute maximum is (0,0)

Rectilinear Motion: motion of an object along line; be able to draw the diagram of an object's motion given the equation for the object's position

x(t) = s(t) = position of the object

x'(t) = v(t) = velocity of the object (if v > 0, then object is moving to the right. v < 0, then the particle is moving to the left. If v = 0, then the object is at rest)

|v(t)| = speed of the object

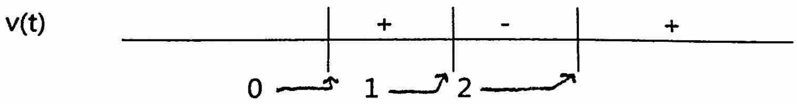
x''(t) = a(t) = acceleration of the object

Procedure:

1. Find v(t) (take the derivative of the position)
2. Find the critical values and make a sign chart
3. Use a table of values to find the position of the object at each critical value.
4. Draw the object's motion on a position graph s(t)

Example: As Kaito dances across the stage, his motion can be modeled by the equation $s(t) = 2t^3 - 9t^2 + 12t - 4$, $t >= 0$. Draw a diagram representing his motion.

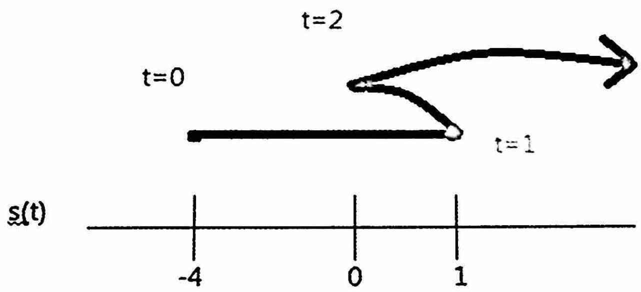
1. $V(t)\ 6t^2 - 18t + 12 = 6(t-1)(t-2)$
2. $t = 1, t = 2$



3.

T	s(t)
0	-4
1	1
2	0

4.

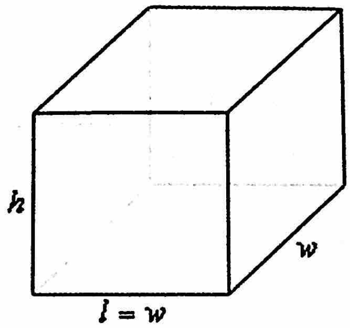


Optimization: find the largest or smallest value of a function

1. Rough sketch
2. Assign variables
3. Write primary equation for the quantity being maximized or minimized
4. Using a secondary equation, change the primary equation to one variable
5. Find the critical value(s) using the first derivative
6. Substitute the critical value(s) into the primary equation to determine the optimized value (be able to state what this means in the context of the problem!)

Example: Assume that the volume of Hatsune Miku is $0.5\ m^3$. Using $10m^2$ of cardboard, we construct a box with the largest volume. What is the largest volume? What are the dimensions that give this volume? How many of Hatsune Miku can we fit in the box?

1. Sketch:



2. V = the maximum volume of the box, l = length of the box, w = width of the box
3. $V = lwh = w^2h$
4. $2lw + 2wh + 2lh = 2w^2 + 4wh = 10$

   $h = \frac{10-2x^2}{4w} = \frac{5-x^2}{2w}$

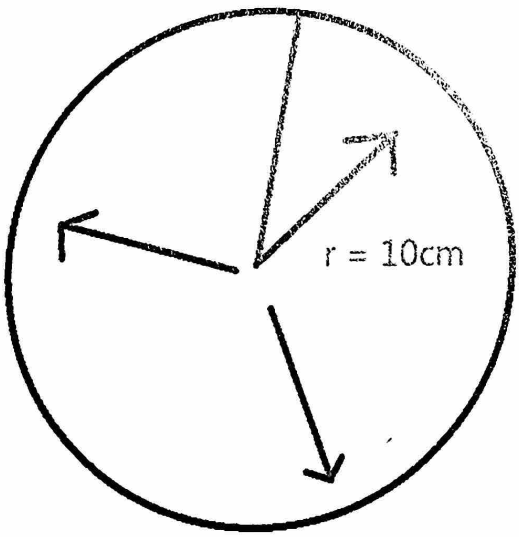
   $V = w^2\frac{5-xw^2}{2w} = \frac{1}{2}(5w\text{-}w^3)$

5. $V'(w) = \frac{1}{2}(5w\text{-}3w^2) = 0$, $w = 0$, $w = \pm\sqrt{\frac{5}{3}}$ (discard $-\sqrt{\frac{5}{3}}$ because width is nonnegative)
6. $V(\sqrt{\frac{5}{3}}) = 2.1517m^3$. The dimensions are $w = \sqrt{\frac{5}{3}}$ m= 1.2910m, l = w = 1.2910 m, $h = \frac{5-w^2}{2w} =$ 1.2910m. We can (technically???) fit 4 of Hatsune Miku in the box ($\frac{2.1517}{0.5} = 4.3$)

Related Rates: the rate at which a quantity changes in relation to the rates of change of other quantities

1. Rough Sketch
2. Assign variables
3. Identify derivatives based on what is given and what is needed
4. Write an equation that relates the rates, using implicit differentiation. You may need to use a secondary equation to convert the given quantities to the quantities needed for the equation, if necessary
5. Solve for the specified quantity (be able to state what this means in the context of the problem!)

Example: A spherical balloon is being inflated with air at a rate of 5cm^3 per minute. At what rate is the radius of the balloon increasing when the radius is 10cm?



1.
2. V = volume of the ballon, r = radius of the balloon

3. $\frac{dV}{dt}$= = 5, $\frac{dr}{dt}$= ??? when r = 10
4. V(t) = $\frac{4}{3}\pi r^3$, so $\frac{dV}{dt}$= $4\pi r^2 \frac{dr}{dt}$
5. $\frac{dV}{dt}$= $4\pi r^2 \frac{dr}{dt}$= 5 = $4\pi(100)\frac{dr}{dt}$, so $\frac{dr}{dt}=\frac{1}{80\pi}$ cm/min (the rate at which the radius of the balloon is increasing)

TOPIC 4: INTEGRAL CALCULUS

Indefinite Integral: the antiderivative of a function that is taken without limits of integration; takes the form $\int f'(x)dx$ = f(x) + c where $f(x)dx$ is the integrand and c is the constant of integration

Indefinite Integral Rules (Basic, Chain, U-Substitution, Parts, and Tabular):

<u>Basic:</u>

$\int adx$ = ax + c   Example: $\int 2dx$ = 2x + c
$\int x^n dx = \frac{x^{n+1}}{n+1}$ + c   Example: $\int x^5 dx = \frac{x^{5+1}}{5+1} + c = \frac{x^5}{6} + c$
$\int af(x)dx$ = a$\int f(x)dx$ where a is a constant   Example: $\int ef(x)dx$ = e$\int f(x)dx$
$\int[f(x) + g(x) + h(x)]dx = \int f(x)dx + \int g(x)dx + \int h(x)dx$
Example: $\int[x^5 + ex + x]dx = \int x^5dx + \int exdx + \int xdx$

<u>Chain Rule:</u>

$\int f(g(x))g'(x)dx$ = $\int f'udu$ where u = g(x)

Example:
$\int x^2(3-10x^3)^4 dx$
u = 3-10x^3, du = -30x^2

the aim of the procedure is to make the part outside the parentheses equal to the derivative of what is inside the parentheses, so in this case, multiply the integrand by -30 and multiply the integral by -1/30

$\int x^2(3-10x^3)^4dx = -\frac{1}{30}\int -30x^2(3-10x^3)^4dx = -\frac{1}{30}(\frac{1}{5})$ (3-10x^3)^5= $-\frac{1}{150}$ (3-10x^3)^5

Special Cases of Chain Rule:

$\int 3(8x-1)e^{4x^2-x}dx$
notice that (8x-1) is the derivative of what is in the exponent of e. all we have to do is move the 3 outside the integral sign (using one of the basic integral rules!)

$\int 3(8x-1)e^{4x^2-x}dx = 3e^{4x^2-x}$ + c

Some other rules to remember:
$\int e^u du$ = e^u + c
$\int a^u du = \frac{a^u}{lna}$ + c
$\int \frac{du}{u}$ = ln $|u|$ + c

U-Substitution (change of Variable): let u replace part of the function and rewrite the function so it is easier to integrate Example:
$\int(2x+2)e^{x^2+2x}dx$
let u = x^2+2x
$\frac{du}{dx}$ = 2x+2 so dx = $\frac{1}{2x+2}$du
$\int(2x+2)e^{x^2+2x}dx = \int(2x+2)\frac{1}{2x+2}e^u du = \int e^u du = e^u$ + c = e^{x^2+2x} +c

Integration by Parts:
$\int u\cdot v' $ = u · v - $\int u'\cdot v$
Example: $\int(x^2 \cdot$ sinx)dx
Let u = x^2 and v' = sinx
u' = 2x, v = -cosx
u · v - $\int u'\cdot v$ = -x^{2}cosx – $\int -2xcosx$ = -x^{2}cosx + $\int \mathbf{2xcosx}$

simplify $\int \mathbf{2xcosx}$
Let u = 2x and v'= cosx
u' = 2, v = -sinx

$\int 2xcosx$ = 2xsinx - $\int 2sinx$ = 2xsinx + 2cosx + c
Therefore:
$\int(x^2 \cdot$ sinx) = -x^{2}cosx + 2xsinx + 2cosx + c

Tabular Integration: given $\int u\cdot v$ where one and exactly one of the terms will eventually reach zero when the derivative is taken an infinite number of times
Example: $\int(x^2 \cdot$ sinx)dx
let u = x^2 and v = sinx

3

construct a table:

u (keep taking the derivative until you reach zero)	v (keep taking the integral until you reach the end of column u)	Alternate between + and – until you reach the end of column u
x^2	sinx	+
2x	-cosx	-
2	-sinx	+
0	cosx	-

multiply diagonally and multiply each product by the sign in the last column:

u (keep taking the derivative until you reach zero)	v (keep taking the integral until you reach the end of column u)	Alternate between + and – until you reach the end of column u
x^2 red	sinx	+ • red
2x green	-cosx	- • green
2 blue	- sinx	+ • blue
0	cosx	-

add the products:

$\int(x^2 \cdot \text{sinx})dx = \underbrace{+x^2(-\cos x)}_{\text{red}}\underbrace{-2x(-\sin x)}_{\text{green}}\underbrace{+2\cos x}_{\text{blue}} + C$

Partial Fractions: sometimes you can rewrite rational functions to find the integral more easily

Example:

$\int(\frac{3x-7}{x^2+5x-6})dx = \int[\frac{A}{x+6} + \frac{B}{x-1}]dx$

solve for A and B:

$\frac{A}{x+6} + \frac{B}{x-1} = \frac{3x-7}{(x+6)(x-1)}$

A(x-1) + B(x+6) = 3x-7

Substitute values of x that will allow you to solve for the constants: (we chose $x=1$ to cancel out A, and $x=-6$ to cancel out B)

If x = 1, then B(1+6) = 3(1)-7, 7B = -4 so B = $-\frac{4}{7}$

If x =-6, then A(x-1) = 3(-6)-7, -A = 3(-6) -7, -7A = -25, so A = $\frac{25}{7}$

Therefore:

$\int[\frac{A}{x+6} + \frac{B}{x-1}]dx = \int[\frac{25}{7(x+6)} - \frac{4}{7(x-1)}]dx = \frac{25}{7}\ln(x+6) - \frac{4}{7}\ln(x-1)$

If the numerator is larger than the denominator (look at the exponents!) then you will need to rewrite the function with polynomial long division:

Example:

$\int(\frac{x^4-5x^3+6x^2-18}{x^3-3x^2})dx$ exponent of 4 is larger than exponent of 3, so...

$$\begin{array}{r l}
 & \underline{x - 2} \\
x^3 - 3x^2 \,\big) & (x^4 - 5x^3 + 6x^2 - 18) \\
 & \underline{-(x^4 - 3x^3)} \\
 & \underline{-2x^3 + 6x^2 - 18} \\
 & -(\underline{-2x^3 + 6x^2) - 18} \\
 & \mathbf{-18}
\end{array}$$

Therefore: $\frac{x^4-5x^3+6x^2-18}{x^3-3x^2} = x - 2 - \frac{18}{x^3-3x^2}$

$\frac{18}{x^3-3x^2} = \frac{18}{x^2(x-3)} = \frac{A}{x} + \frac{B}{x^2} = \frac{C}{(x-3)}$ solve for A, B, and C, and then integrate as before (note the repeated factor of x)

??? **Definite Integral Properties:** see pg 26

$\int_a^b f(x)dx = \lim_{\Delta x\to 0}(\sum_{i=1}^{n} f(c_i)^n \cdot \Delta x_i$ where a and b are the limits of integration

$\int_a^a f(x)dx = 0$ if f is defined at x = a Example: $\int_a^a (4x^6)dx = 0$

$\int_b^a f(x)dx = -\int_a^b f(x)dx$ if f is integrable on [a,b] Example: $\int_2^4 4x^6dx = -\int_4^2 4x^6dx$

$\int_b^a f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ if f is integrable on [a,b] and a<c<b

Example: $\int_2^4 4x^6dx = \int_2^3 4x^6dx + \int_3^4 4x^6dx$

$\int_b^a kf(x)dx = k\int_b^a f(x)dx$ Example: $\int_2^4 4x^6dx = 4\int_2^4 x^6dx$

Fundamental Theorem of Calculus Parts I and II:

Fundamental Theorem of Calculus Parts I: if f(x) is a continuous function on the closed interval [a,b] and F(x) is an antiderivative of f(x) on [a,b], then:

$\int_a^b f(x)dx = F(b)-F(a)$

Example:

$\int_0^5 x^2dx = (\frac{1}{3})x^3 \Big|_3^5 = (\frac{1}{3})5^3 - (\frac{1}{3})0^3 = \frac{125}{3}$

Fundamental Theorem of Calculus Part II: if f is a continuous function on the open interval I containing a, then for every x in the interval:

$\frac{d}{dx}\int_a^x f(t)dt = f(x)$

Example 1: $\frac{d}{dx}\int_0^x \sqrt{t^2+1}dt = \sqrt{x^2+1}$ (in this case, simply replace t with x, the upper limit of integration)

Note: for the above procedure to work, the lower limit of integration must be a constant and the upper limit of integration must be x.

Example 2:

$\frac{d}{dx}\int_0^{x^3} cost dt = ???$

$\cos'(t) = \frac{cos'(t)}{du} \cdot \frac{du}{dx} = \frac{d}{du}(\int_0^{x^3} cost dt) \cdot \frac{du}{dx}$

let u = x^3

$\frac{d}{du}(\int_0^{x^3} cost dt) \cdot \frac{du}{dx} = \frac{d}{du}(\int_0^{u} cost dt) \cdot \frac{du}{dx} = \cos u(3x^2)$ (fundamental theorem of calculus part II)
$= (\cos x^3)(3x^2)$ (in this case, replace t with the upper limit of integration AND multiply by the derivative of the upper limit of integration)

Note: for the above procedure to work, the lower limit of integration must be a constant

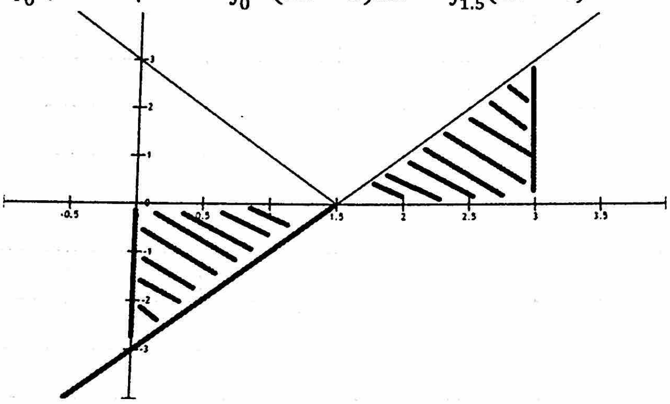
Example 3:

$\frac{d}{dx}\int_x^{2x} \sin(t^2)dt = ???$

Write as the sum of integrals and integrate each part separately, using the fundamental theorem of calculus part II

$\int_x^{2x} \sin(t^2)dt = \int_x^0 \sin(t^2)dt + \int_0^{2x} \sin(t^2)dt = -\int_0^x \sin(t^2)dt + \int_0^{2x} \sin(t^2)dt$
$= -\sin(x^2) + 2\sin(4x^2)$

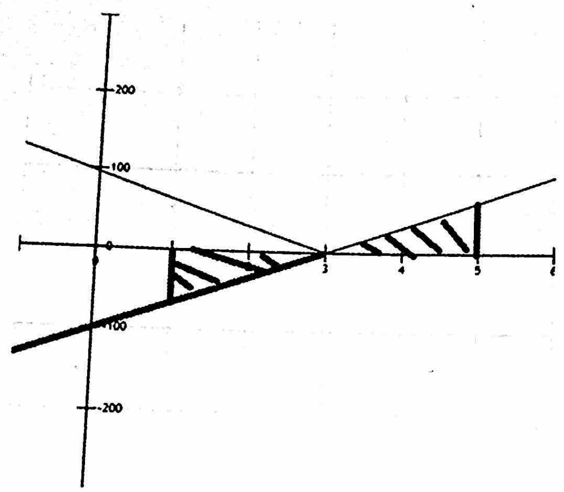
Absolute Value: using the definition of absolute value, rewrite the integral as the sum of two integrals
Example: f(x) = |2x-3|
$\int_0^3 |2x-3|dx = -\int_0^{1.5}(2x-3)dx + \int_{1.5}^3 (2x-3)dx$



<u>Net distance (displacement):</u>
$\int_a^b v(t)dt$
<u>Total distance:</u>
$\int_a^b |v(t)|dt$
Example: Gumi throws a baseball upward with an initial velocity of 96 ft/sec. The velocity of the ball, in ft/sec, can be approximately modeled by the function v(t) = 96-32t. Find:

a) the net displacement of the ball from t = 1 to t=5
b) the total displacement of the ball from t = 1 to t=5

net distance:
$\int_1^5 (96-32t)dt = 96t - (16t^2) \Big|_1^5 = [96(5) - (16)(25)] - [96(1) - (16)(1)] = 80 - 80 = 0$ ft
total distance:
$\int_1^5 |96-32t|dt = -\int_1^3 (96-32t)dt + \int_3^5 (96-32t)dt = 64 + 64 = 128$ ft



Reimann Sums: for a closed interval [a,b], where Δx is a partition of [a,b], where a= $x_0 < x_1 < x_2 < x_3 ... < x_{n-1} < x_n = b$, where Δx_i is the width of the *i*th subinterval. Let ci be any point in the ith subinterval, then the sum for the partition Δx:

$$\sum_{i=1}^{n} f(c_i)\Delta x,\ x_{n-1} \le c_i \le x_i \text{ (Riemann Sum)}$$

$$\int_x^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\frac{b-a}{n} \text{ (for even subintervals)}$$

Right Hand Riemann Sum: use the right-most subdivision of each interval

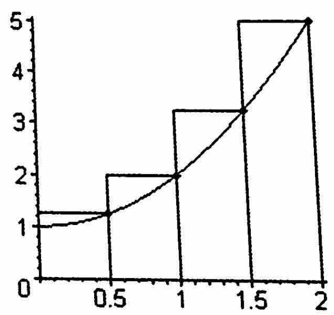
$f(x_0)\Delta x + f(x_1)\Delta x ... f(x_{n-1})\Delta x$ (evaluate f(x) at the left endpoint of each subinterval)

Example: a=0, b=2, n=4 (there are four rectangles)

(b-a)/n = 2/4 = 0.5

The right-end divisions of the subinterval are: 0.5, 1, 1.5, and 2.

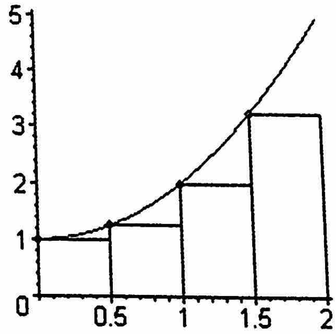
$f(0.5)\Delta x + f(1)\Delta x + f(1.5)\Delta x + f(2)\Delta x \approx 1.2(0.5) + 2(0.5) + 3.2(0.5) + 5(0.5) = 5.7$



Left Hand Riemann Sum: use the left-most subdivision of each interval

The left-end divisions of the subinterval are: 0, 0.5, 1, and 1.5.

$f(0)\Delta x + f(0.5)\Delta x + f(1)\Delta x + f(1.5)\Delta x \approx 1(0.5) + 1.2(0.5) + 2(0.5) + 3.2(0.5) = 3.7$

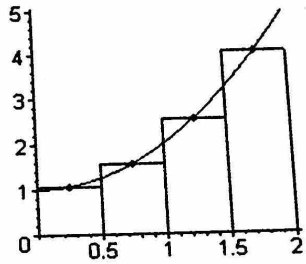


Midpoint Riemann Sum: use the midpoint of each interval (take the average of the two boundaries of each subdivision)

Example:

The midpoints are: $\frac{0+0.5}{2} = 0.25, \frac{1+0.5}{2} = 0.75, \frac{1+1.5}{2} = 1.25, \frac{1.5+2}{2} = 1.75$

$f(0.25)\Delta x + f(0.75)\Delta x + f(1.25)\Delta x + f(1.75)\Delta x \approx (1.05)(0.5) + 1.6(0.5) + 2.6(0.5) + 4.2(0.5) = 4.725$



Trapezoidal Riemann Sum: let $y_0, y_1, y_2 \ldots y_n$ be the bases of the trapezoids, and let the heights be $h_1, h_2, h_3 \ldots h_n$

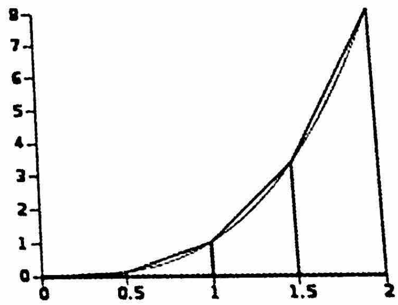
$S = \frac{y0+y1}{2}h_1 + \frac{y1+y2}{2}h_2 + \frac{y2+y3}{2}h_3 \ldots \frac{y_n+y_{n-1}}{2}h_n$

If the subintervals are the same length, then:

$T(n) = \frac{h}{2}(y_0 + 2y_1 + 2y_2 \ldots + 2y_{n-1} + y_n)$

Example:

$S = \frac{0+0.5}{2}0.1 + \frac{0.5+1}{2}1 + \frac{1+1.5}{2}3.5 \ldots \frac{1.5+2}{2}8 = 19.15$



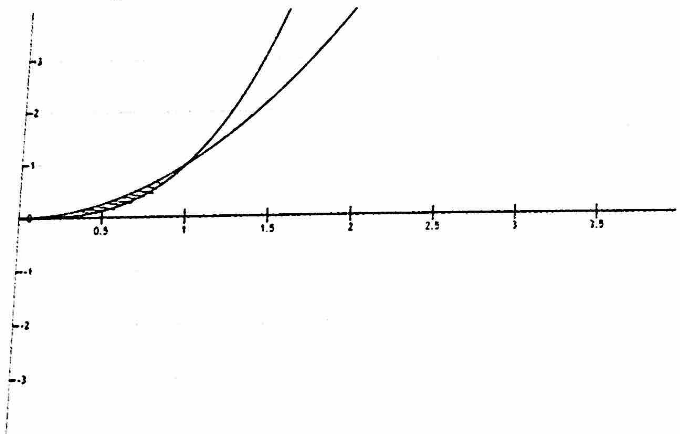
Area Between Two Curves:

$A = \int_a^b [f(x) - g(x)]dx$ where f(x) is the upper function and g(x) is the lower function

$A = \int_c^d [g(y) - f(y)]dy$, where g(y) is the rightmost function and f(y) is the leftmost function

Example 1: $f(x) = x^2$, $g(x) = x^3$ (a quick sketch shows us that f(x) is the upper function)

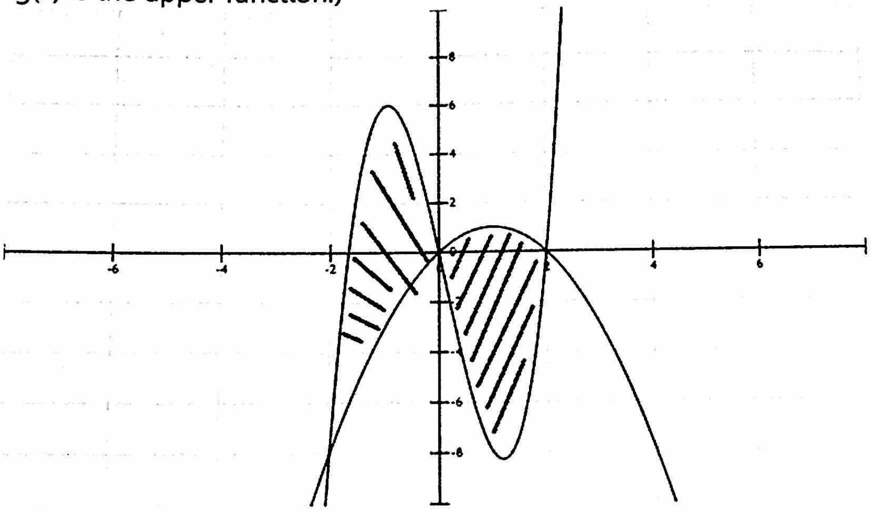
$A = \int_a^b [f(x) - g(x)]dx = \int_0^1 [x^2 - x^3]dx = \frac{1}{3}x^3 - \frac{1}{4}x^4 \Big|_0^1 = \frac{1}{12}$



Example 2: $f(x) = 3x^3-x^2-10x$, $g(x) = -x^2+2x$

A = $\int_a^b [f(x) - g(x)]dx = \int_{-2}^{0} [f(x) - g(x)]dx + \int_0^2 [g(x) - f(x)]dx$

(two integrals are necessary here because on the left side, f(x) is the upper function, and on the right side, g(x) is the upper function.)



Volume of Known Solids of Revolution: (volume created when a function is revolved around an axis. same rules apply regarding the order of the functions, and writing two integrals if necessary, see previous example)

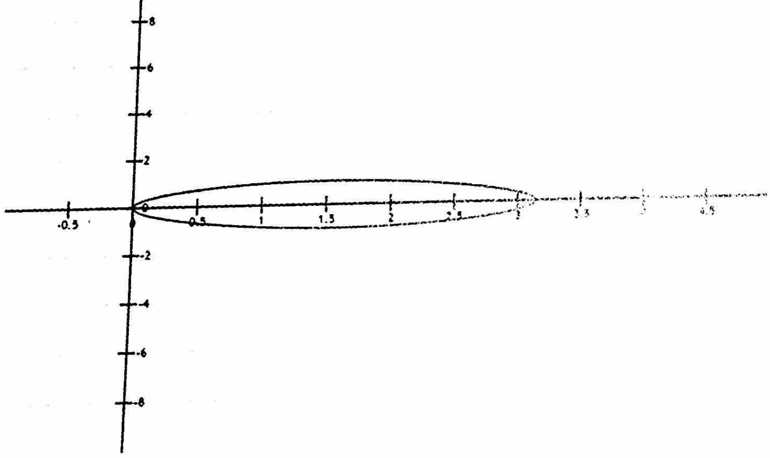
For one function:

V = $\pi \int_a^b f(x)^2 dx$ (revolve around x-axis)

V = $\pi \int_c^d f(y)^2 dy$ (revolve around y-axis)

Example: $f(x) = \sqrt{\sin x}$

V = $\pi \int_0^{\pi} (\sqrt{\text{sinx}})^2 dx = \pi$ (-cos) $|_0^{\pi} = 2\pi$



For two functions:

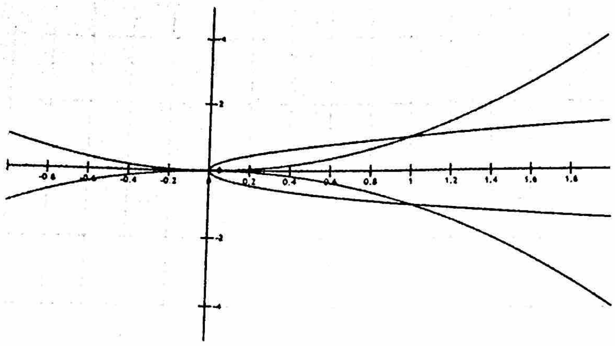
V = pi$\int_a^b f(x)^2 - g(x)^2 dx$ (revolve around x-axis)

V = pi$\int_a^b g(y)^2 - f(y)^2 dx$ (revolve around y-axis)

Example:

$F(x) = \sqrt{x}$, $g(x) = x^2$

V = $\pi \int_0^1 (\sqrt{x})^2 - (x^2)^2 dx = \pi[\frac{1}{2}x^2 - \frac{1}{5}x^5] \,|_0^1 = \frac{3}{10}\pi$



Revolving around the line y = L:

V = $\pi \int_a^b (f(x) - L))^2 - (g(x) - L))^2\, dx$ (where x = L)

V = $\pi \int_a^b (g(y) - L))^2 - (f(y) - L))^2\, dy$ (where y = L)

Example: x = y^2, x = √y, rotated about the line y = 1

V = $\pi \int_0^1 (y^2 - 1)^2 - (\sqrt{y} - 1)^2\, dy$

Volume of Known Solids with Known Cross-Sections: t = top function, b = bottom function. "slices" are perpendicular to the x-axis

squares: V = $\int_a^b (t - b)^2$ dx

equilateral triangle: V = $\sqrt{\frac{3}{4}} \int_a^b (t - b)^2$ dx

semi-circle: V = $\frac{\pi}{8} \int_a^b (t - b)^2$ dx

isosceles right triangle:

leg: V = $\frac{1}{2} \int_a^b (t - b)^2$ dx

hyp: V = $\frac{1}{4} \int_a^b (t - b)^2$ dx

rectangle: cross section is a rectangle whose height is # times the length of its base

V =# $\int_a^b (t - b)^2$ dx

Example: Let R be the region bounded by y = sin(πx) and y = x^3-4x. R is the base of a solid, where each of its cross section perpendicular to the x-axis is a square. Write an expression for the volume of this solid.

y = sin(πx) is the upper function, so:

$$V = \int_0^2 (\sin(\pi x) - (x^3 - 4x))^2\, dx$$

Area of Surface of Revolution: where r(x) is the distance between the graph of x and the axis of revolution (function minus axis of revolution)

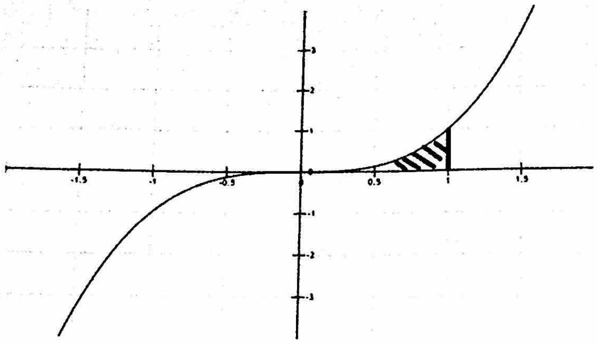
S = $2\pi \int_a^b r(x) \cdot \sqrt{(1 + f'(x))^2}$ dx

S = $2\pi \int_a^b r(x) \cdot \sqrt{(1 + g'(y))^2}$ dy

Example: Let f(x) = x^3 [0,1]

F'(x) = $3x^2$

S = $2\pi \int_a^b r(x) \cdot \sqrt{(1 + (3x^2))^2}$ dx ≈ 3.562



Average Value:

If f is integrable on [a,b], then average value = $\frac{1}{b-a}\int_a^b f(x)dx$

Example:

f(x) = $\frac{9}{x^3}$ [1,3]

AV = $\frac{1}{3-1}\int_1^3 \frac{9}{x^3}\,dx = \frac{1}{2}\cdot\left(-\frac{9}{2x^2}\right)\Big|_1^3$ = 2 (the average y-value is 2)

Mean Value Theorem:

If f is continuous on [a,b] then there exists a number c in [a,b] such that $\int_a^b f(x)dx$ dx = f(c)(b-a)

Example:

2 = $\frac{9}{x^3}$

x = $\sqrt[3]{\frac{9}{2}} \approx 1.651$

Arc Length: for a smooth curve beginning at (a,c) and ending at (b,d) and a<b, c<d, then

$l = \int_a^b \sqrt{(}1+\left(\frac{dy}{dx}\right)^2)$

Example:

f(x) = x^3[0,1]

$l = \int_0^1 \sqrt{(}1+(3x^2)^2) \approx 1.548$

TOPIC 5: PARAMETRIC EQUATIONS

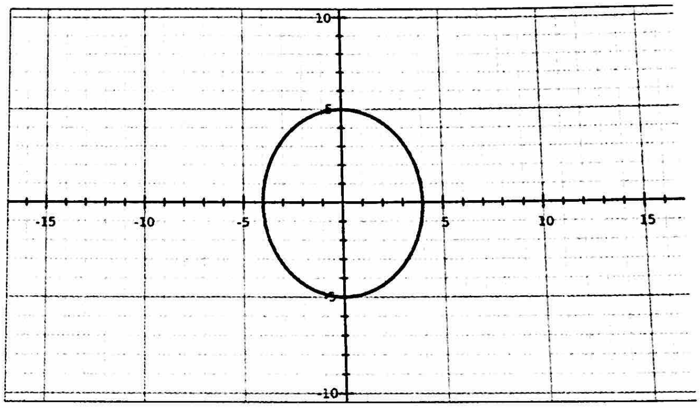
Parametric Function: if the x coordinates and y coordinates of a point on a graph are given by the functions $x = f(t)$, $y = g(t)$, then t is the parameter. parametric equations are often used to represent the motion of a particle, with t representing time

Eliminating the Parameter: combine the two parametric equations into one equation (thus eliminating t)

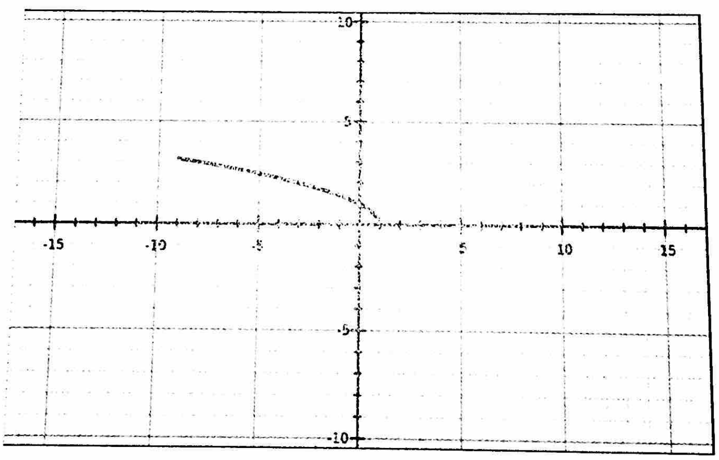
Example 1:
$x = 4\sin(t)$, $y = 5\cos(t)$

$\sin(t) = \frac{x}{4}$ $\cos(t) = \frac{y}{4}$

$\sin^2(t) + \cos^2(t) = \left(\frac{x}{4}\right)^2 + \left(\frac{y}{5}\right)^2 = 1$



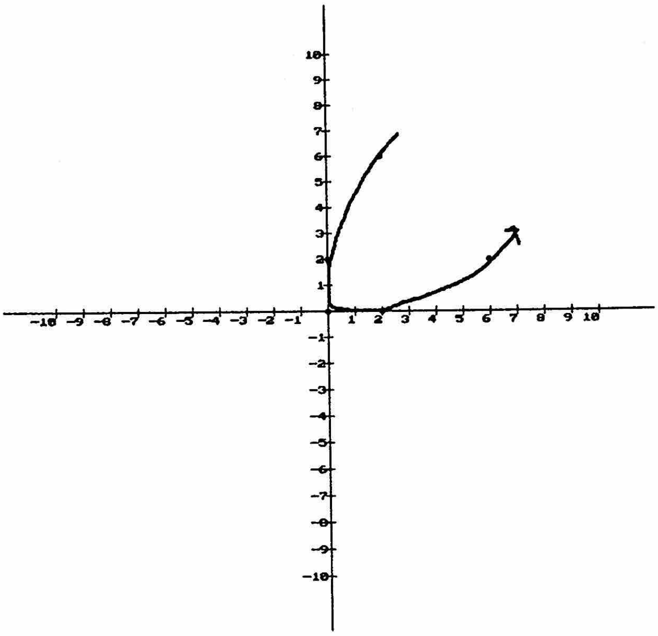
Example 2: $x = 1 - t$, $y = \sqrt{x}$
$y^2 = t$
$x = 1-y^2$
important note: because the domain of $y = \sqrt{x}$ is $[0,\infty)$, the domain of $x = 1-y^2$ is restricted to a domain of $(-\infty, 1]$ (because $1-y^2$ is always less than or equal to 1 if y^2 is always greater than or equal to 0)



Sketching: at various values of t, evaluate both parametric equations and plot the points. if t is time, you can draw an arrow indicating the direction of motion

Example: $x = t^2 + t$, $y = t^2 - t$

t	x	y
-2	2	6
-1	0	2
0	0	0
1	2	0
2	6	2



Finding the First and Second Derivative of a Parametric Equations:

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ and $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$

Example: x = 2sinθ, y = cos(2θ)

$\frac{dx}{d\theta} = 2\cos\theta$, $\frac{dy}{d\theta} = -2\sin 2\theta$, $\frac{dy}{dx} = \frac{-2\sin 2\theta}{2\cos\theta} = -\frac{2\sin\theta\cos\theta}{\cos\theta} = -2\sin\theta$

$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left(\frac{dy}{dx}\right)}{\frac{dx}{d\theta}} = \frac{-2\cos\theta}{2\cos\theta} = -1$

Arc Length for Parametric Equations:

$L = \int_{\alpha}^{\beta} \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]}dt$ where $\alpha \le t \le \beta$ (Important note: if the curve is traced out multiple times, take the limits of integration for one tracing only)

Example: x = 3sin(t), y = 3cos(t), $0 \le t \le 2\pi$

$\frac{dx}{dt} = 3\cos(t)$ $\frac{dy}{dt} = -3\sin(t)$

$L = \int_0^{2\pi} \sqrt{[9\sin^2 t + 9\cos^2 t]}dt = 6\pi$

TOPIC 6: DIFFERENTIAL EQUATIONS

Separation of Variables, Growth and Decay, and Logistics:

<u>Separation of Variables:</u> to solve an equation involving a derivative, move like variables together and integrate, using the initial conditions to solve for the constant of integration. If you are not given the constant of integration, you can still find the general solution:

Example 1: Find the general solution of $\frac{dy}{dx} = \frac{x}{y}$

$\frac{dy}{dx} = \frac{x}{y}$

$ydy = xdx$

$\int ydy = \int xdx$

$\frac{1}{2}y^2 = \frac{1}{2}x^2 + c$

$y^2 = x^2$

Example 2: Solve $xe^{x^2} + yy' = 0$, given the initial condition that y = 1 when x = 0

$y\frac{dy}{dx} = -xe^{x^2}$

$ydy = -xe^{x^2}dx$

$\int ydy = \int -xe^{x^2}dx$

$\frac{y^2}{2} = -\frac{1}{2}e^{x^2} + c$

$y^2 = -e^{x^2} + c$

$1 = -e^{0^2} + c$, $c = 2$, so: $y^2 = -e^{x^2} + 2$

<u>Growth and Decay:</u> let N = the number of members in the population, t is time, k is a proportionality constant

$\frac{dN}{dt}$ **= kN**

$\frac{dN}{N} = kdt$ (separation of variables)

$\int \frac{dN}{N} = \int k$

$\ln N = kt + C$

$\mathbf{N} = e^{kt+c} = e^{kt}e^{c} = \mathbf{Ae^{kt}}$ where $A = e^{c}$

Logistics: logistic growth occurs when population growth is limited (limit = L, which is also known as the carrying capacity)

$\frac{dy}{dt}$ **= ky(1 -** $\frac{y}{L}$**)**

$\int \frac{dy}{y(1-\frac{y}{L})} = \int kdt$

$\int (\frac{1}{y} + \frac{1}{L-y})$ dy = $\int$ kdt (rewriting the left side with partial fractions)

ln|y| - ln|L-y| = kt + c

ln$|\frac{L-y}{y}|$ = -kt – c (multiply each side by -1)

$|\frac{L-y}{y}|$ = e^{-kt-c} = $(e^{-c})e^{-kt}$

$\frac{L-y}{y}$ = be^{-kt} where b = ± e^{-c}

y = $\frac{L}{1+be^{-kt}}$

Word Problems:

Example 1: Let v(t) be the velocity (ft/sec) on a skydiver at time t seconds, where t ≥ 0. After her parachute opens, her velocity satisfies the differential equation $\frac{dv}{dt}$ = -2v – 32, with the initial condition v(0) = 50. Find an expression for v in terms of t.

$\frac{dv}{dt}$ = -2v – 32 move v together with dv (keep -2 on the right side):

$\frac{dv}{v+16}$ = (-2)(dt) integrate both sides:

$\int \frac{dv}{v+16} = \int (-2t)(dt)$

ln|v+16| = -2t+c

$e^{\ln|v+16|} = e^{-2t+c}$

v= e^{-2+c} -16

$v = C'e^{-2t} - 16$ where $C' = e^c$
$-50 = C'e^0 - 16$
$C' = -34$

$v = -34e^{-2t} - 16$

Example 2: Let $y = Ce^{kt}$ represent the number of fruit flies in a sample. If there were 100 flies after the second day of the experiment and 300 flies after the fourth day, what was the original population?

Based on the information in the problem, you can write the two equations:

$100 = Ce^{2k}$ $300 = Ce^{4t}$

Solve for C in the first equation:

$100 = Ce^{2k}$
$C = 100e^{-2k}$

$300 = Ce^{4t} = 100e^{-2k}e^{4t} = 100e^{2k}$
$3 = e^{2k}$
$\ln 3 = 2k$
$k = \frac{1}{2}\ln 3 \approx 0.549$

$y = Ce^{kt} = 100e^{-2(0.549)} \approx 33$

Example 3:

Example 3: Let the initial population for some elk be 40. After 5 years, the population is 104 elk. If the maximum capacity of the population is 4000, what will be the elk population in 15 years?

$y = \frac{L}{1+be^{-kt}} = \frac{4000}{1+be^{0}} = 40$
$\frac{4000}{1+b} = 40$, $b = 99$

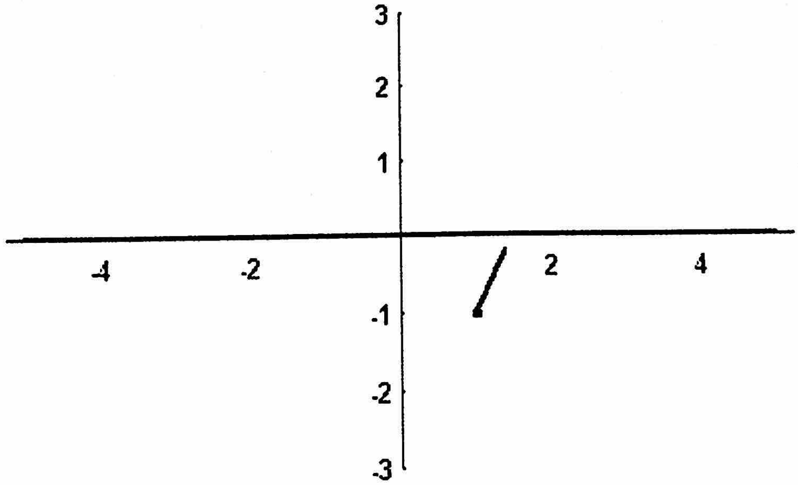
$104 = \frac{4000}{1+99e^{-5k}}$ $k \approx .194$

$y = \frac{4000}{1+99e^{-.194t}} = \frac{4000}{1+99e^{-.194(15)}} \approx 626$

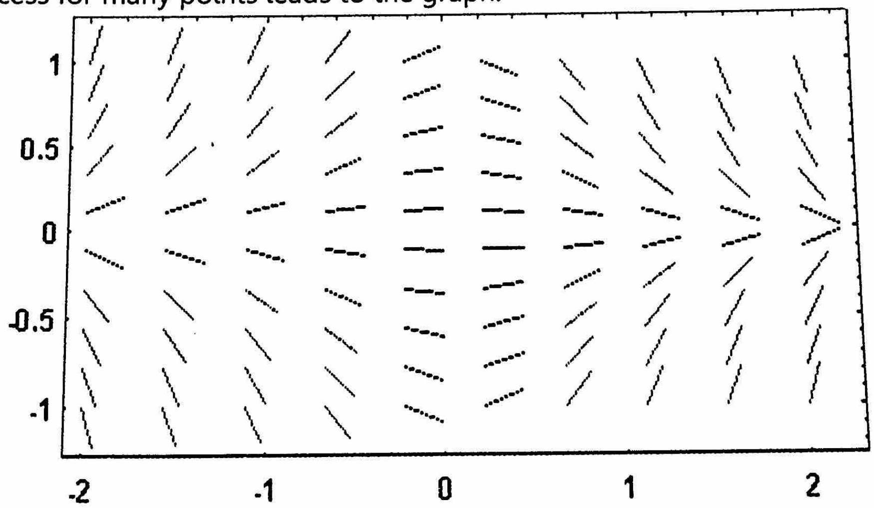
Slope Fields: visual representation of the solutions for first-order differential equations

Example: $\frac{dy}{dx} = -2xy$
To find the slope field line at (1, -1) for example, substitute x = 1 and y = 1 into the equation.
$\frac{dy}{dx} = -2(1)(-1) = 2$

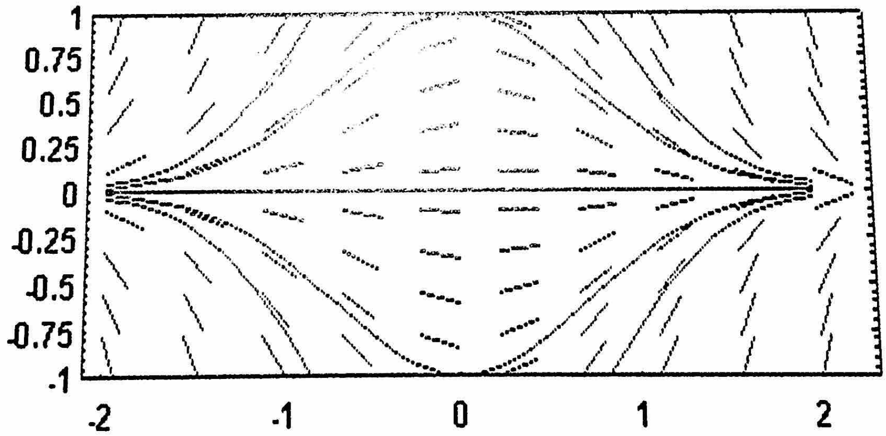
In the x-y plane, graph a segment with a slope of 2, at (1,-1) to represent this solution.



Repeating this process for many points leads to the graph:

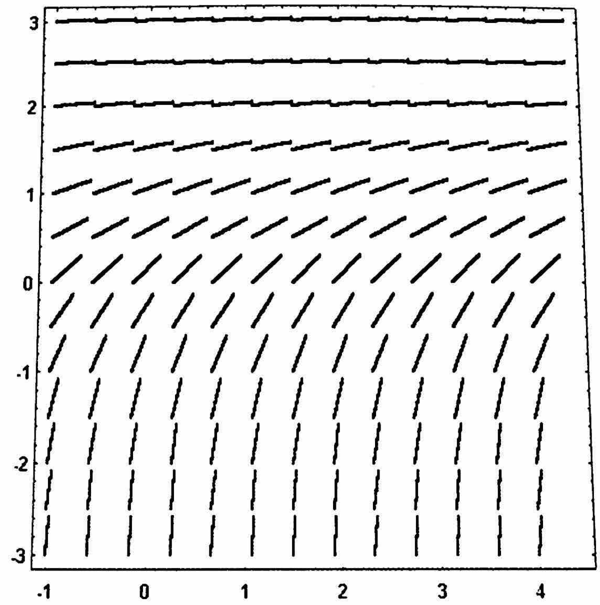


Treating the line segments as tangent lines to the graph, sketch possible solutions for the function:



Each black line represents one solution to the differential equation $\frac{dy}{dx}$ = -2xy for various values of c. Of course, you may be given a slope field graph and asked to match to a differential equation. In this case, test points in the x-y field to see which differential equation it matches.

Example: Does this slope field graph match the differential equation $\frac{dy}{dx}$ = e^{-x} or $\frac{dy}{dx}$ = e^{-y}?



As y increases, the slope decreases, whereas changes in x have no effect on the slope. The slope field matches $\frac{dy}{dx}$ = e^{-y}.

Euler's Method: numerical method for solving differential equations for specific values, using tangent line approximation. Approximate x value = $(x+\Delta x)$, Approximate y value = $(y+\Delta y)$

Example: solve $\frac{dy}{dx}$ = y for y = 2, given the initial condition that y = 1 when x = 0. Use a step size of Δx =1.

Construct a table:

x	y	$\frac{dy}{dx}$	$\Delta y = m\Delta x$
0 (initial condition)	1 (initial condition)	1 (equation definition)	$\Delta y = (1)(1) =$ **1**
1 $(x+\Delta x)$	**1**+**1** = **2** $(y+\Delta y)$	2	$\Delta y = (2)(1) =$ **2**
2 $(x+\Delta x)$	**2** + **2** = **4** (solution)		

TOPIC 7: VECTOR VALUED FUNCTIONS

r(t) = f(t)$\mathbf{i}$ + g(t)$\mathbf{j}$ = $\langle f(t), g(t)\rangle$
r(t) = f(t)$\mathbf{i}$ + g(t)$\mathbf{j}$ + h(t)$\mathbf{k}$ = $\langle f(t), g(t), h(t)\rangle$
where f(t), g(t), and h(t) are component functions

Domain, Sketching, and Limits:

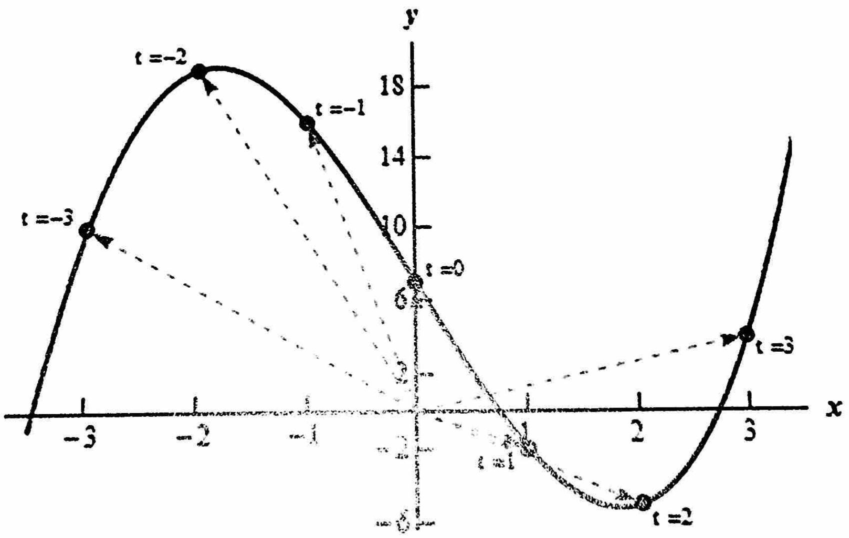
Domain: the set of all real numbers (t) for which all the component functions are defined

Example: the domain of $\vec{r}$ = $\langle \cos(t), \ln(4-t), \sqrt{t+1}\,\rangle$ is [-1,4) because cos(t) is defined for all t's, the second component is defined for t < 4, and the third component is defined for t ≥ -1.

Sketching: evaluate the function at several values of t, then graph the corresponding vectors and connect the endpoints in ascending order, according to t

Example: $\vec{r}$ = $\langle t, t^3 - 10t + 7\rangle$

$\vec{r}$ (-3) = $\langle -3,10\rangle$, $\vec{r}$ (-2) = $\langle -2,19\rangle$, $\vec{r}$ (-1) = $\langle -1,16\rangle$, $\vec{r}$ (0) = $\langle 0,7\rangle$, $\vec{r}$ (1) = $\langle 1,-2\rangle$, $\vec{r}$ (2) = $\langle 1,-5\rangle$, $\vec{r}$ (-3) = $\langle 1,-2\rangle$



Limits:

$\lim_{t\to a} r(t)$ = $\lim_{t\to a} f(t)$i + $\lim_{t\to a} g(t)$j
$\lim_{t\to a} r(t)$ = $\lim_{t\to a} f(t)$i + $\lim_{t\to a} g(t)$j + $\lim_{t\to a} h(t)$k

Example: $\lim_{t\to 0} r(t)$ = e^ti + $\frac{\sin t}{t}$ = 1$\mathbf{i}$ + 1$\mathbf{j}$ = <1,1>

Velocity and Acceleration Vectors for Motion on a Plane Curve:

v(t) = $\frac{dr}{dt}$ = $\langle f'(t), g'(t), h'(t)\rangle$

a(t) = $\frac{d^2r}{dt^2}$ = $\langle f''(t), g''(t), h''(t)\rangle$

$\int r(t)dt = \int f(t)dt\, \boldsymbol{i} + \int g(t)dt\, \mathbf{j} + \int h(t)dt\, \mathbf{k}$ (each integral on the right side will yield its own constant of integration, for which you must solve using the initial condition

Example 1: The motion of an object can be described with the vector valued function r(t) = $\langle t^2, e^{2t}, t^3\rangle$. Find the velocity and acceleration of this object.

v(t) = $\langle 2t, 2e^{2t}, 3t^2\rangle$
a(t) = $\langle 2, 4e^{2t}, 6t\rangle$

Example 2: A particle moves along the x-axis with a velocity of v(t) = 1 – sin(2πt). Find the position x(t) of the particle at t, given that x(0) = 0.

x(t) = $\int v(t)dt = \int [1 - \sin(2\pi t)]dt$ = t + $\frac{1}{2\pi}\cos(2\pi t)$ + C
0 = $\frac{1}{2\pi}$ cos(0) + C
C = - $\frac{1}{2\pi}$
x(t) = t + $\frac{1}{2\pi}\cos(2\pi t)$ - $\frac{1}{2\pi}$

Example 3: Evaluate $\int r'(t) = \int \sin(2t)\, \boldsymbol{i} - \int 2\cos(t)\, \boldsymbol{j} + \int \frac{1}{1+t^2}\, \mathbf{k}$, given that r(0) = 3**i** -2**j** + k

$\int r'(t)$ = (- $\frac{1}{2}$cos2t + c_1)i – (2sint + c_2)j + (arctan(t) + c_3)k
- $\frac{1}{2}$cos2t + c_1 = 3, $c_1 = \frac{7}{2}$
– (2sint + c_2) = -2, c_2 = 2
(arctan(t) + c_3) = 1, c_3 = 1

$\int r'(t)$ = (- $\frac{1}{2}$cos2t + $\frac{7}{2}$)i – (2sint +2)j + (arctan(t) +1)k

Speed: ||v(t)|| = ||r(t)|| = $\sqrt{((x'(t))^2 + ((y'(t))^2}$

Example: An object's motion can be described by the vector valued function r(t) = (t^2-4)**i** + ($\frac{1}{2}$t)**j**. Calculate its speed.

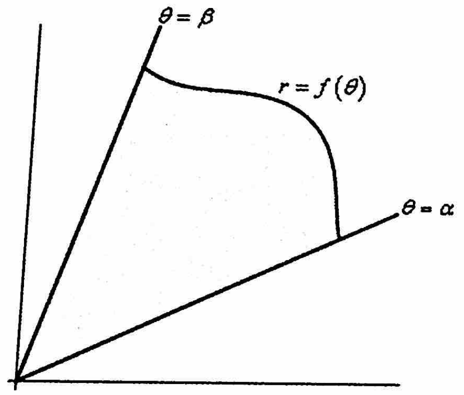
||v(t)|| = $\sqrt{(2t)^2 + \left(\frac{1}{2}\right)^2}$ = $\sqrt{4t^2 + \frac{1}{4}}$

TOPIC 8: POLAR CURVES

Area Bounded by Polar Curves (One Curve, Two Curves, Shared, In One Out the Other):

<u>One Curve:</u>

A = $\int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta$ where α and β are the angle values for which the desired area begins and ends

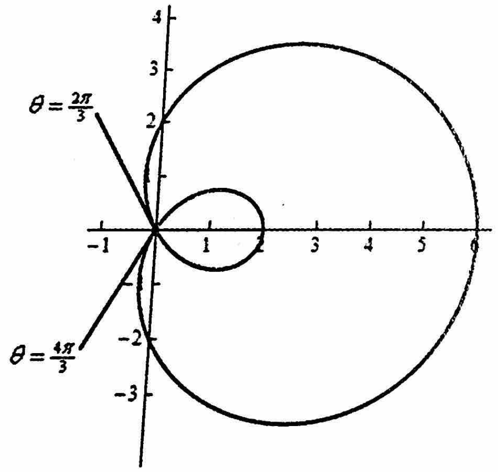


Example: Determine the area in the inner loop of r = 2+4cosθ
You can determine the values of α and β by setting 2+4cosθ equal to 0.

$0 = 2 + 4\cos\theta$

$\cos\theta = -\frac{1}{2}$

$\theta = \frac{2}{3}\pi, \frac{4}{3}\pi$



$$A = \int_{\frac{2}{3}\pi}^{\frac{4}{3}\pi} \frac{1}{2}(2 + 4\cos\theta)^2 d\theta = \int_{\frac{2}{3}\pi}^{\frac{4}{3}\pi} \frac{1}{2}(4 + 16\cos\theta + 16\cos^2\theta)d\theta = \int_{\frac{2}{3}\pi}^{\frac{4}{3}\pi}(2 + 8\cos\theta + 4(1 + \cos(2\theta))\, d\theta$$

$$= \int_{\frac{2}{3}\pi}^{\frac{4}{3}\pi}(6 + 8\cos\theta + 4\cos(2\theta))\, d\theta = (6\theta+8\sin\theta+2\sin(2\theta))\Big|_{\frac{2}{3}\pi}^{\frac{4}{3}\pi} = 4\pi - 6\sqrt{3} \approx 2.17$$

Two Curves:

$A = \int_{\alpha}^{\beta} \frac{1}{2} (r_0^2 - r_i^2)\, d\theta$ where r_0 is the outer curve and r_i is the inner curve

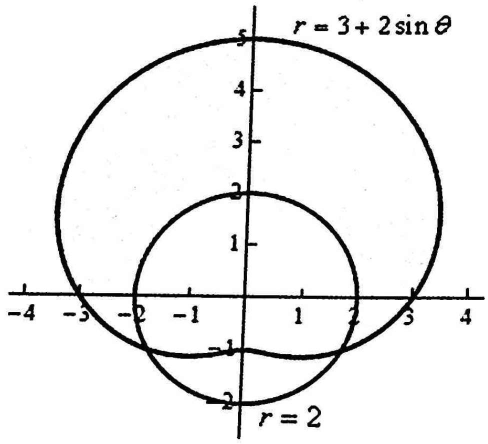
Example: Determine the area inside r = 3 + 2sinθ and outside r = 2.

3 + 2 sinθ = 2 (the area does not begin from r = 0!)

$\sin\theta = -\frac{1}{2}$

$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

$A = \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{1}{2} (r_0^2 - r_i^2)\, d\theta = \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{1}{2} ((3 + 2\sin\theta)^2 - 2^2)\, d\theta \approx 24.19$



Shared:

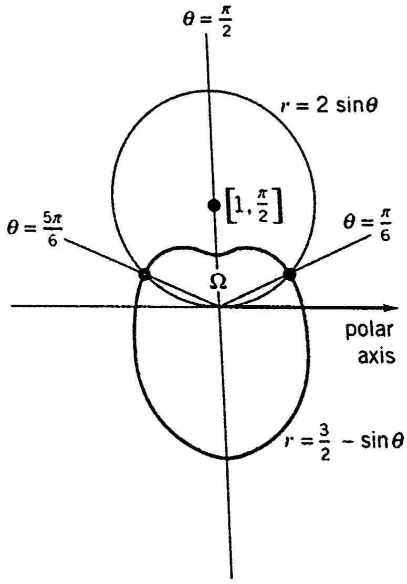
$A = \int_{\alpha}^{\beta} \frac{1}{2} r_0^2 d\theta$ where r_0 is the outer curve (this may change over the interval, so you will need to write a separate integral for each time the outer curve changes!)

Example: Determine the area between r = 2sinθ and $r = \frac{3}{2} - \sin\theta$

Intersection points: $2\sin\theta = \frac{3}{2} - \sin\theta$

$\sin\theta = \frac{1}{2}$, so $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$A = \int_0^{\frac{\pi}{6}} \frac{1}{2} (2\sin\theta)^2 d\theta + \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2} (\frac{3}{2} - \sin\theta)^2 d\theta + \int_{\frac{5\pi}{6}}^{\pi} \frac{1}{2} (2\sin\theta)^2 d\theta = \frac{5\pi}{4} - \frac{15}{8}\sqrt{3}$

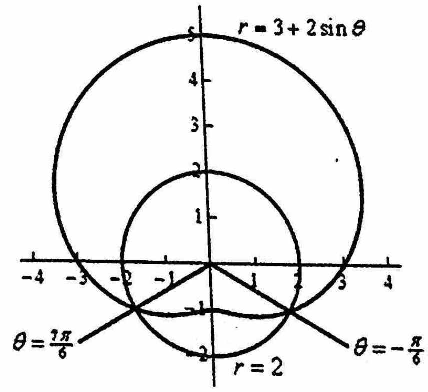


Inside One, Outside the Other:

$A = \int_{\alpha}^{\beta} \frac{1}{2} (r_0^2 - r_i^2)\, d\theta$ where r_0 is the outer curve and r_i is the inner curve

Example: Determine the area outside r = 3 + 2sinθ and inside r = 2.

$A = \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{1}{2} (2^2 - (3 + 2\sin\theta)^2)\, d\theta \approx 2.2$



Slope of tangent (dy/dx): If r = f(θ), then:

x = f(θ)cosθ
y = f(θ)sinθ

$\frac{dx}{d\theta}$ = f′(θ)cosθ – f(θ)sin θ (using the product rule for derivatives)
$\frac{dy}{d\theta}$ = f′(θ)sinθ + f(θ)cosθ (using the product rule for derivatives)

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

Example: Evaluate $\frac{dy}{dx}$ for r = cos(2θ)

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-2\sin\theta\cos\theta + \cos(2\theta)\sin\theta}{-2\sin(2\theta)\sin\theta + \cos(2\theta)\cos\theta}$$

Substituting a value for θ will give the slope of the line tangent to the curve at that value.

Arc Length: L = $\int_{\alpha}^{\beta}(r^2 + (\frac{dr}{d\theta})^2)d\theta$
Example: Determine the length of r = θ [0,1]
L = $\int_0^1(\theta^2 + (1)^2)d\theta \approx 1.15$

TOPIC 9: SEQUENCES AND SERIES

Improper Integrals (Continuous and Discontinuous):

Continuous Functions:

if f is continuous on [a,∞), then:

$$\int_a^{\infty} f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$$

if f is continuous on (-∞,a), then:

$$\int_{-\infty}^{a} f(x)dx = \lim_{b\to-\infty} \int_b^a f(x)dx$$

If f is constant (-∞,∞), then:

$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_c^{\infty} f(x)dx$, c ~~(doublebar epsilon)~~ $\in \mathbb{R}$

Example: $\int_0^{\infty} e^{-\frac{x}{2}}\,dx = \lim_{b\to\infty} \int_0^b e^{-\frac{x}{2}}dx = \lim_{b\to\infty}[(-2e^{-\frac{x}{2}}) \mid_0^b] = \lim_{b\to\infty}[(-2e^{-\frac{b}{2}}) - (-2)] = 2$ (converges to 2)

Discontinuous Functions:

if f is continuous on [a,b), then:

$\int_a^b f(x)dx = \lim_{c\to b^-} \int_a^c f(x)dx$

if f is continuous on (a,b}, then:

$\int_a^b f(x)dx = \lim_{c\to a^+} \int_c^b f(x)dx$

If f is continuous on [a,c) U (c,b], then:

$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Example: $\int_{-1}^{0} \frac{1}{x^2}$ dx = $\lim_{c\to 0^-} \int_{-1}^{c} \frac{1}{x^2}dx = \lim_{c\to 0^-} \left[-\frac{1}{x}\right] \mid_{-1}^{c} = = \lim_{c\to 0^-} \left[-\frac{1}{c} - 1\right] = \infty - 1 = \infty$ (diverges)

Sequences (Definition, Limits, Writing):

Definition: function whose domain is the set of positive integers
Infinite series: a_1, a_2, a_3... are the terms of the sequence
a_n: how the sequence is defined (the nth term), sometimes written in brackets {}
Limit: let f be a function of a real variable such that $\lim_{x\to\infty} f(x) = L$
If an is a sequence such that f(n) = an, for every positive integer n, then $\lim_{n\to\infty} a_n = L$ (what the sequence converges to)

If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$

Writing Sequence, given Definition: for the first term, substitute n = 1 into the definition. for the second term, substitute n = 2 into the definition, and so forth.

Example: $a_n = 3 + (-1)^n = 3 + (-1)^1, 3 + (-1)^2, 3 + (-1)^3, 3 + (-1)^4... = 2,4,2,4...$

Writing Definition, given Sequence: identify patterns in the terms and write a definition for the nth term

Example: 24, 16, 8, 0, -8...
Each term is 8 less than the previous term
$a_n = x-8n$
substitute 24 (the first term) for a_n and 1 for n to find x
$24 = x - 8(1)$
$X = 32$
Therefore: $a_n = 32-8n$

Definition of Series:

if a_n is a sequence of real numbers, then an infinite series is defined by:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 \ldots + a_n + \cdots$$

Sequences of Partial Sums: the sum of part of a sequence (S)

Example:

$$\sum_{n=1}^{4} n^2 = 1^2 + 2^2 + 3^2 + 4^2 = S_4 = 30$$

Properties of Partial Sums: if

$\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$ and $c \in \mathbb{R}$

then:

$$\sum_{n=1}^{\infty} c \cdot a_n = c \cdot A$$

$$\sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n = A \pm B$$

$$\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$$

Example:

$$\sum_{n=1}^{\infty} \left(\frac{1}{4n} + \frac{1}{5^n}\right) = \sum_{n=1}^{\infty} \left(\frac{1}{4n}\right) + \sum_{n=1}^{\infty} \left(\frac{1}{5^n}\right) = \frac{1}{4} \sum_{n=1}^{\infty} \left(\frac{1}{n}\right) + \sum_{n=1}^{\infty} \left(\frac{1}{5^n}\right)$$

$\sum_{n=1}^{\infty} \left(\frac{1}{5^n}\right)$ is a geometric series with a = $\frac{1}{5}$ and r = $\frac{1}{5}$, so S = $\frac{1}{4}$

$\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)$ is a p-series that diverges

If one part diverges, the sum must diverge as well!

Tests for Convergence and Divergence:

removing a finite number of terms from the beginning of a series does not affect convergence and divergence

nth term test for divergence:

if $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges

if $\lim_{n\to\infty} a_n = 0$, then the test is inconclusive

Example:

$\sum_{n=1}^{\infty} \frac{n}{2n+1}$ diverges because $\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} \neq 0$

geometric series test: if a = 0, then a + ar + ar^2 + ar^3... is a geometric series
if |r| < 1 then the series converges (S = $\frac{a}{1-r}$)
if |r| ≥ 1, then the series diverges

Example:
$\sum_{n=1}^{\infty}\left(\frac{2}{5}\right)^n$ converges because $|\frac{2}{5}| < 1$. Its sum is $\frac{\frac{2}{5}}{1-\frac{2}{5}} = \frac{5}{3}$

p-series test:

$$\sum_{n=1}^{\infty}\frac{1}{n^p}$$

Converges when p>1
Diverges when 0<p≤1

Example:
$\sum_{n=1}^{\infty}\frac{1}{n}$ diverges because p≤1
$\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges because p>1

integral test: if f(x) is a continuous, positive, and decreasing function, and f(n) = a_n, then $\sum a_n$ converges if and only if $\int_1^{\infty} f(x)dx$ converges:

Example: $\sum_{n=1}^{\infty}\frac{n}{n^2+1}$ diverges because
$\int_1^{\infty}\frac{x}{x^2+1}dx = \lim_{b\to\infty}\frac{1}{2}\ln(x^2+1)\Big|_1^b = \infty$

comparison test:

If $\sum a_n$ converges and an > bn, then $\sum b_n$ converges
If $\sum a_n$ diverges and an < bn, then $\sum b_n$ diverges

Example: $\sum_{n=1}^{\infty}\left(\frac{1}{n^n}\right)$
By comparing $\frac{1}{n^n}$ with $\frac{1}{2^n}$, we see that $\sum_{n=1}^{\infty}\left(\frac{1}{n^n}\right)$ converges because $\frac{1}{2^n}$ converges (geometric series, r = $\frac{1}{2}$)
and $\frac{1}{n^n} < \frac{1}{2^n}$ for n > 2

limit comparison test:

if $\lim_{n\to\infty}\frac{a_n}{b_n}$ is finite and positive, then $\sum a_n$ and $\sum b_n$ both converge or diverge

Example:
$\sum_{n=1}^{\infty}\left(\frac{1}{2n+1}\right)$
Let $a_n = \frac{1}{2n+1}$ and $b_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\frac{1}{2n+1}}{\frac{1}{n}} = \frac{1}{2}$

We know that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-series) so $\sum_{n=1}^{\infty} \left(\frac{1}{2n+1}\right)$ must diverge as well

alternating series test: ($a_n > 0$)

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 \dots + (-1)^n a_n + \cdots$$

Converges if $a_{n+1} < a_n$ for all n and $\lim_{n\to\infty} a_n = 0$

Absolute convergence: $\sum a_n$ converges absolutely if $\sum |a_n|$ converges

Conditional convergence: if $\sum |a_n|$ diverges and $\sum a_n$ converges then $\sum a_n$ converges conditionally

Example 1:

$\sum \frac{(-1)^n n^2}{n^2+9}$ diverges because $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{n^2+9} = 1 \neq 0$

Example 2: $\sum \frac{\sin(\frac{1}{3}n\pi)}{n^2}$

$\lim_{n\to\infty} \frac{\sin(\frac{1}{3}n\pi)}{n^2} = 0$ and $\frac{\sin(n\pi)}{n^2} > \frac{\sin([n+1]\pi)}{(n+1)^2}$

$\sum \left|\frac{\sin(\frac{1}{3}n\pi)}{n^2}\right|$ converges (comparison test with $\frac{1}{n^2}$)

So $\sum \frac{\sin(\frac{1}{3}n\pi)}{n^2}$ converges absolutely

Example 3: $\sum (-1)^n \frac{1}{n}$

$\lim_{n\to\infty} \frac{1}{n} = 0$ and $\frac{1}{n} > \frac{1}{n+1}$

$\sum \frac{1}{n}$ diverges (p-series), so $\sum (-1)^n \frac{1}{n}$ converges conditionally

ratio test:

if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$ exists, then $\sum a_n$ converges if $L < 1$ and diverges if $L > 1$. If $L = 1$, the test is inconclusive

Example:

$\sum_{n=1}^{\infty} \frac{1}{n!}$

$$\lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{(n+1)} = 0 < 1$$ so the series converges

TOPIC 10: TAYLOR SERIES

Taylor Series Centered at x = a:

$$f(x) \approx \frac{f(a)}{0!}\cdot(x-a)^0 + \frac{f'(a)}{1!}\cdot(x-a)^1 + \frac{f''(a)}{2!}\cdot(x-a)^2 + \frac{f'''(a)}{3!}\cdot(x-a)^3 + \ldots \frac{f^n(a)}{n!}\cdot(x-a)^n$$

if a = 0, then it becomes a Maclaurin Series:

$$f(x) \approx \frac{f(0)}{0!}\cdot(x)^0 + \frac{f'(0)}{1!}\cdot(x)^1 + \frac{f''(0)}{2!}\cdot(x)^2 + \frac{f'''(0)}{3!}\cdot(x)^3 + \ldots \frac{f^n(0)}{n!}\cdot(x)^n$$

Example: Write the third degree Taylor polynomial centered at x = 1 for f(x) = √x

$f(1) = 1$

$f'(1) = (\frac{1}{2})(x^{\frac{-1}{2}}) = \frac{1}{2}$

$f''(1) = (-\frac{1}{2})(\frac{1}{2})(x^{\frac{-3}{2}}) = -\frac{1}{4}$

$f'''(1) = (-\frac{1}{4})(-\frac{3}{2})(x^{\frac{-5}{2}}) = \frac{3}{8}$

$$f(x) \approx \frac{1}{0!}\cdot(x-1)^0 + \frac{\frac{1}{2}}{1!}\cdot(x-1)^1 + \frac{-\frac{1}{4}}{2!}\cdot(x-a)^2 + \frac{\frac{3}{8}}{3!}\cdot(x-1)^3$$

$$= 1 + \frac{1}{2}\cdot(x-1)^1 - \frac{1}{8}\cdot(x-1)^2 + \frac{1}{16}\cdot(x-1)^3$$

Common Maclaurin Series:

$e^x \approx 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \ldots \frac{1}{n!}x^n$

$\sin x \approx x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \ldots \frac{(-1)^n}{(2n+1)!}x^{2n+1}$

$\cos x \approx 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \ldots \frac{(-1)^n}{(2n)!}x^{2n}$

$\frac{1}{1-x} \approx 1 + x + x^2 + x^3 + x^4 + \ldots x^n$

Manipulation of Taylor Series (Differentiation, Antidifferentiation, Forming New Series):

Diffrentiation:

if $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 \ldots$

then $f'(x) = \sum_{n=0}^{\infty} n\cdot a_n (x-a)^{n-1} = a_1 + 2a_2(x-a) + 3a_3(x-a)^2$

Antidiffrentiation:

if $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 \ldots$

then $\int f(x) = c + \sum_{0}^{\infty} a_n \frac{(x-a)^{n+1}}{n+1} = c + a_0(x-a) + a_1\frac{(x-a)^2}{2} + a_2\frac{(x-a)^3}{3} \ldots$

Formation of New Series from Known Series: Manipulate a known series to write an unknown one

Example: f(x) = $\frac{1}{1-x}$ = 1 + x + x^2 + x^3 + x^4 + ... x^n

g(x) = $\frac{1}{1+x}$ replace x with (-x) in the first polynomial to obtain the polynomial approximation for g(x)

g(x) = $\frac{1}{1+x}$ = f(-x) = $\sum_0^\infty (-1)^n (x)^n$

h(x) = - $\frac{1}{(1+x)^2}$ = g'(x) (use the rule for diffrentiation)

h(x) = $\sum_0^\infty (-1)(n)(x)^{n-1}$

Functions defined by Power Series:

power series in x:

$\sum_0^\infty a_n x^n$= a$_0$ + a$_1$x + a$_2$x^2 + ... $a_n x^n$

power series in (x-a)

$\sum_0^\infty a_n(\mathrm{x}-\mathrm{a})^n$= a$_0$ + a$_1$(x-a) + a$_2$((x-a))^2 + ... $a_n((\mathrm{x}-\mathrm{a}))^n$

Modifications: same principles for manipulation apply

Example: f(x) = e^x = 1 + x + $\frac{1}{2}$x^2 + $\frac{1}{6}$x^3 + $\frac{1}{24}$x^4 + ... $\frac{1}{n!}$x^n

g(x) = e^{2x} = f(2x) = 1 + (2x) + $\frac{1}{2}$(2x)^2 + $\frac{1}{6}$(2x)^3 + $\frac{1}{24}$(2x)^4 + ... $\frac{1}{n!}$(2x)^n

When a modification is performed, the radius of convergence does not change, but you will still need to test the endpoints (see next section)

Radius and Interval of Convergence:

Interval ~~Radius~~ of convergence: set of all values for x for which the power series converges

Radius of convergence: if the power series converges when |x-a| < r and diverges when |x-a| > r then r is the radius of convergence

For a power series $\sum_0^\infty a_n(\mathrm{x}-\mathrm{a})^n$, only one of the following will be true:
- the power series converges for all values of x (interval of convergence: ℝ; radius of convergence: ℝ)
- the power series converges only where centered at x = a (interval of convergence: {a}, radius of convergence: 0)
- the power series converges for all x in an open interval but not outside a closed interval (interval of convergence: [(a±R)], radius of convergence: ℝ)

To find the radius of convergence, use the ratio test. Then, substitute each endpoint into the series to determine the interval of convergence.

Example 1:

$\sum_{n=1}^\infty \frac{x^n}{n!}$

$\lim_{n\to\infty} |\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}| = \lim|\frac{|x|}{n+1}| = 0 < 1$

Radius of convergence: ℝ

Interval of Convergence: $\mathbb{R}$

Example 2:

$\sum_{n=1}^{\infty} n!\, x^n$

$\lim_{n\to\infty} (n+1)x = \infty$

Radius of convergence: 0

Interval of Convergence: {0}

Example 3:

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x)^{n-1}}{n+1}$

$\lim_{n\to\infty} |\frac{x^n}{n+2} \cdot \frac{n+1}{x^{n-1}}| = \lim|x| < 1$

Radius of convergence is 1

Interval of Convergence:

When x = 1, the series converges (alternating series test)

When x = -1, the series diverges (comparison test with $\frac{1}{n}$)

So the interval of convergence is (-1, 1]

Lagrange Error Bound for Polynomials:

f(x) ≈ P(x)

f(x) = Pn(x) + R(x) where R(x) is the Lagrange Remainder

If you use a Taylor polynomial of degree n with center c, to approximate x, then the actual function falls within the error bound:

$R_n(x) = E_{max}$ <, $|\frac{f^{n+1}(z)(x-c)^{n+1}}{(n+1)!}|$ where z is a "mystery number" such that c ≤ z ≤ x and $f^{n+1}(z)$ is maximized, and x is the value we are trying to approximate

Example: Find the Lagrange Error Bound for the 3rd degree Maclaurin polynomial approximation of f(x) = cos(x) for x = 0.1

n = 3

c = 0

x = .1

0 ≤ z ≤ .1

$F^4(z)$ is maximized when z = 0 (cosx is decreasing on the interval)

$R_3(x) = E_{max} \leq |\frac{f^4(0)(0.1)^4}{(4)!}|$